LSF Research Working Paper Series

N°. 17-10

Date: November 2017

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JEL classification: G14, D82, D83
Keywords: Order Book, Market Depth, Limit Order Execution Rate, Welfare

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Inefficient Market Depth*

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November 22, 2017

Abstract

An investor who uses a limit order in order to trade, instead of a market order, saves the bid-ask spread but incurs an execution delay. Thus, the use of limit orders slows down the rate at which gains from trade are realized, and then has a negative effect on welfare. With comparative statics, I show how some liquidity measures co-vary with investors' welfare. I find that market depth negatively co-varies with welfare while the limit order execution rate positively co-varies with welfare. Indeed, when market depth is due to orders inefficiently queuing in the book, the limit order execution rate is low. It suggests that limit order execution rate should be taken into consideration for assessing market quality.

*I gratefully acknowledge helpful comments and suggestions from Jean-Edouard Colliard, Thierry Foucault, Austin Gerig and participants at the Frontiers of Finance 2014 Conference, the ESSFM Gersenzee evening sessions, the Banque de France seminar.

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1 Introduction

Investors' welfare is one main objective of financial market regulators. At first glance, a liquid market is synonym of large welfare gains for investors. The standard definition for market liquidity is the ease at which a given quantity of an asset can be traded, at a price that does not deviate much from a benchmark. Put differently, market liquidity corresponds to some implicit trading costs. In centralized markets, these implicit trading costs are usually measured with bid-ask spreads and market depths (the number of quotes closed to the best bid and ask prices). These liquidity measures are biased toward the welfare of liquidity consumers. However, most centralized markets (stock, FX,... etc) are organized as limit order markets in which any investor can trade by supplying liquidity with limit orders. Market depth, or bid-ask spread, do not account well for the welfare of liquidity suppliers. For instance, a high market depth may arise because of a long time-to-execution after limit order submission, which would not seem beneficial for limit order users. How is market depth determined by the trading strategies of investors? How can it be linked to investors' welfare?

To address these questions, I use a dynamic model of limit order market à la Dugast (2017). I provide closed-form solutions for equilibrium outcomes such as market depth, trading volume, limit order execution rate, and welfare. Based on a comparative statics exercise, I find that (i) market depth negatively co-varies with welfare, (ii) limit order execution rate positively co-varies with welfare , and (iii) in most cases trading volume positively co-varies with welfare, except for a specific range of parameters. It shows, first, that limit orders execution rate and market depth may vary in opposite directions and, second, that execution quality of limit orders may dominate in the welfare outcome. The corollary is that one cannot infer investor's welfare variations from variations in market depth only. Consequently, the limit order execution rate should also be looked upon when assessing market quality.

In my model I consider a continuous-time framework. There is a continuum of competitive investors who can hold 0 or 1 unit of an asset. Each investor has either a high or low private value for the asset. The asset private value of an agent is random and idiosyncratic. It follows a two-state continuous-time Markov chain and switches from high to low, or conversely, with same intensity. The difference in asset valuations across agents generates motives for trade and welfare gains when some asset shares are transferred from investors with a low private value to investors with a high private value. Trading takes place vis an order book in which any investor can supply liquidity with limit orders or consume liquidity with market orders.

My paper focuses on steady-state equilibria, that is such that, on aggregate, the limit order book does not change over time. At equilibrium, the bid and the ask prices are constant over time and the bid-ask spread is equal to the tick size, the minimal difference between two available trading prices. All limit orders are submitted at the best bid and ask prices. Buy (resp. sell) limit orders are submitted by investors with a high private value who do not own the asset (resp. with a low private value who own the asset). The number of limit orders on each side of the book, the market depth, is such that investors are indifferent between using a limit order, to trade at a good price but with a time delay, or a market order to trade immediately with an implicit cost, the bid-ask spread. When the tick size decreases, the comparative advantage of using limit orders declines. The maximal time delay for limit order execution, that investors are willing to bear, declines as well. It implies that the market depth decreases and that investors use relatively more market orders than limit orders.

Welfare is negatively linked to the market depth. Ideally, from the perspective of a social planner, any investors who is waiting in the book to have his limit order executed would be matched with a similar investor on the other side of the book. The trade between two of these investors would transfer the asset from a low value type to a high value type and thus would increase welfare. The tick-size of the market allows investors to use limit orders to extract more of the trading surplus than their counterpart with market orders, without risking to have their limit orders undercut by other investors. This relative "market power" that is offered to liquidity suppliers is inefficient since it slows down trading and the subsequent trading surplus realization. Hence the tick-size has a negative impact on welfare.

In the empirical literature, market depth is used to assess market quality. For instance, Chordia, Roll and Subrahmanyam (2000, 2001) study aggregate movements and co-movements of liquidity for NYSE stocks that, at the time, were run by specialists (market-makers). Among different liquidity proxies, they use quoted bid-ask spreads and quoted depths. In theses markets, quoted prices and offered quantities were announced by specialists prior to a trade. In this case, liquidity proxies precisely reflect implicit trading costs that final investors were facing when making the decision to trade. In limit order markets, one can proceed to similar exercises and study market depth, with order book data (e.g., Biais, Hillion and Spatt (1995), Engle, Fleming, Ghysels and Nguyen (2011), Hasbrouck and Saar (2013)). However, trading costs are money transfers from liquidity consumers to liquidity suppliers. It prevents from drawing conclusion about market quality/welfare. Intuitively, welfare should be high when the frequency at which gains from trade are realized is high as well, as shown in Colliard and Foucault (2012). Generally speaking, it is not clear how this trade frequency should be linked to market depth and bid-ask spread.

The paper is organized as follows. Section 2 presents the setup and assumptions of the model. Section 3 describes the model equilibrium. Section 4 derives model outcomes. Section 5 analyzes welfare implications. Section 6 concludes.

2 Model

The model directly builds on Dugast (2017). This section presents the model's four building blocks: the asset and its payoff structure, investors' preferences, the investors' market contacts, and the trading mechanism. I also define investors' trading strategies.

2.1 The Asset pay-off

I consider the market for a risky asset in a continuous time framework. The asset can be traded during the time interval, $[0, \tau_{end}]$, where the termination date, τ_{end}, is random and follows a Poisson process of intensity, r. At date τ_{end}, the asset pays off, [1], v per share.

2.2 Investors' preferences

There is a unit mass of risk-neutral investors who can trade the asset. As in Duffie, Gârleanu, and Pedersen (2005, 2007), an investor is characterized by her private valuation of the asset, which can be high or low. An investor with a low private valuation continuously incurs a holding cost δ per asset unit; an investor with a high private valuation incurs no such cost. At each instant, an investor's private valuation may switch from high to low - or from low to high - at a Poisson intensity ρ.

The case on which I focus is that where the initial distribution of preferences is the stationary one where half of the population has a high private valuation of the asset while

[1] This initial setup is equivalent to Duffie, Gârleanu, and Pedersen (2005); in that paper, the horizon is infinite, investors are infinitely lived and have a time discount rate r, and the asset value v is a utility flow for the investor who holds one share. Returning to my setup requires only that one replace v with v/r.

the other half has a low private valuation. In aggregate, then, the distribution of preferences remains unchanged throughout the game.

Asset supply and holding constraint. As in Duffie, Gârleanu, and Pedersen (2005, 2007), I assume that an investor can own either one unit or zero units of the asset. I assume also that the asset supply is equal to 1/2. Therefore, half of the population owns the asset at any particular time.[2]

Investors' types. In the rest of this paper, an investor's *type* is defined as the combination of his asset-holding status and his private valuation of the asset. Each investor's type belongs to the set $\{ho, hn, lo, ln\}$, where h and l denote "high" and "low" and where o and n signify "owner" and "non-owner". Hence the mass of investors can be divided into four populations: L_{ho}, L_{hn}, L_{lo}, and L_{ln}. Because there is a unit mass of investors, these four populations must sum to unity:

$$L_{ho} + L_{hn} + L_{lo} + L_{ln} = 1. \tag{1}$$

Since the asset supply is 1/2, it follows that the mass of asset owners is - at any moment - half of the unit mass of investors:

$$L_{ho} + L_{lo} = \frac{1}{2}. \tag{2}$$

Finally, the mass of high-private valuation investors is, at any point in time, equal to 1/2,

$$L_{ho} + L_{hn} = \frac{1}{2}. \tag{3}$$

Overall, there are three independent equations, (1), (2), and (3), and four unknowns, L_{ho}, L_{hn}, L_{lo} and L_{ln}. To determine the four populations at equilibrium, there remains one free parameter to pin down. Proposition 1 derives this free parameter, which is denoted α_t.

Proposition 1. *There is an $\alpha \in \mathbb{R}$ such that the different masses of population satisfy*

$$L_{hn} = L_{lo} = \alpha, \text{ and } L_{ho} = L_{ln} = \frac{1}{2} - \alpha. \tag{4}$$

α_t must satisfy the constraints of non-negativity for the populations, $\frac{1}{2} - \alpha \geq 0$, and $\alpha \geq 0$.

[2]Setting the asset supply to 1/2 renders the problem symmetric between buyers and sellers of the asset. By a continuity argument, the results reported here should hold for any supply $s \in [0, 1]$ with s "close enough" to 1/2.

At this stage, the trading mechanism has not been specified. The equilibrium value of α results from additional building blocks.

2.3 Market contacts

Whenever an investor's private valuation changes, she *contacts* the market. The following assumption precisely defines a market contact.

Assumption 1. *An investor contacts the market when the investor' private valuation changes, at Poisson intensity ρ. An investor who contacts the market (i) observes the state of the market and the asset value, and (ii) can take an action.*

Assumption 1 may seem restrictive and deserves therefore some explanations. If investors could contact the market more often than only for private valuation changes (for instance continuously) then the equilibrium outcome would be the same. Investors find it optimal to trade only when their private valuation changes.

2.4 The Limit Order Book

Trading takes place via a limit order book. All traders' quotes must be posted on a grid. The minimum difference between two prices is the tick size, Δ. Investors can use limit or market orders to trade. A limit order specifies a limit price at which the order can be executed, and that order remains posted in the order book until it is matched with a market order or is cancelled. Since each order is for a single share, the *depth* of the limit order book at price P, denoted D_P, is the number of limit orders posted at price P. Market orders do not specify a price limit, so they are executed immediately at the best quote on the opposite side.

Assumption 2. *The price grid is the set $\mathcal{P} = \{n\Delta,\ n = 0, 1, 2, \ldots\}$.*

Assumption 3. *Limit orders are executed according to a "random matching" rule. In particular: when a market order hits the book, all limit orders submitted at the same price have the same probability of being executed - that is, regardless of their respective submission dates.*

Assumption 3 simplifies the analysis because two investors of the same type and with a limit order in the book at the same price will have the same value function.

In practice, of course, time priority does apply. Dugast (2017) discusses how the model can be solved with time priority and show that the main economic mechanisms remain unchanged.

That being said, time priority does not apply across trading platforms in fragmented markets. Thus the real-world matching of the aggregate flow of market and limit orders may reflect a mix of time priority and random matching.

Assumption 4. *The holding cost, δ, and the tick size, Δ, are such that*

$$\delta > (r + 2\rho)\Delta \tag{5}$$

Assumption 4 states that δ is large when compared with Δ. This assumption ensures that the gain from trade, which has an order of magnitude of δ, is large enough compared to the bid-ask spread, which in equilibrium is equal to the tick size Δ. Investors prefer not to trade when those relative sizes are reversed.

2.5 Investors' Trading Strategies

I now define an investor's action set and strategy as well as the equilibrium concept.

Action set. Whenever an investor contacts the market, she can take actions. These actions can be any combination of *elementary actions*, though they are performed under two constraints.

- As an owner, the investor can take only the following elementary actions: do nothing and remain an owner; submit a sell limit order; submit a sell market order and become a non-owner, if there is a buy limit order in the order book; or cancel her previous sell limit order.

- As a non-owner, the investor can take only the following elementary actions: do nothing and remain a non-owner; submit a buy limit order; submit a buy market order and become an owner, if there is a sell limit order in the order book; or cancel her previous buy limit order.

- The constraints on the action set are that the investor (i) own either zero or one asset share and (ii) have at most one limit order in the order book.

An investor can, in theory, play any sequence of elementary actions. However, in equilibrium and under constraints (i) and (ii), strategies will involve short and intuitive sequences of elementary actions.

Strategy. An investor's strategy consists of a mapping to the action set from the investor's history and current type, the current state of the market, and the clock time. A formal definition is provided in the Online Appendix (Section A).

Equilibrium concept. The focus in this paper is on *Markov perfect equilibria*,[3] in which an investor's strategy depends only on the investor's current type, and on the current aggregate state of the order book.

3 Equilibrium

In this section I describe the paper's focal steady state equilibrium.

3.1 Steady state and one-tick market

Here I begin by defining the notion of market in steady state. I then show that, in a steady-state phase, the bid-ask spread must be equal to one tick in equilibrium.

Definition 1. *The market is in steady state when the aggregate state of the order book (trading prices, market depths) and the aggregate order flow (submissions, executions, cancellations) are deterministic and constant over time.*

During a steady state phase, the best bid and ask prices, B and A, do not change and market depths at these prices, D_A and D_B, remain constant. Even though the order book looks always the same in aggregate, trading occurs in equilibrium. Moreover, the order flow is constant. For instance, at each instant t, the same flows (i.e. quantities per infinitesimal time interval dt) M_A of buy market orders and M_B of sell market orders are submitted. The following proposition shows that, when trading occurs on both sides of the market - that is, $M_A > 0$ and $M_B > 0$ - the bid-ask spread must be equal to one tick.

Proposition 2. *If the market is in steady state in an equilibrium in which trading occurs on both sides of the market at each instant (i.e., $M_A > 0$ and $M_B > 0$), then its bid-ask spread is equal to one tick, i.e., $A - B = \Delta$.*

[3] Dugast (2017) verifies that the "one-shot deviation principle" applies: a strategy admits no profitable deviation if and only if it admits no profitable deviation that lasts for "one period".

Proof. Consider an investor who finds it optimal to send a buy market order at price A; the existence of such an investor is guaranteed because trading occurs on both sides of the market. If there is a price P on the grid such that $B < P < A$, then the investor would be strictly better-off sending a limit order at P. Indeed, he expects his order to be immediately hit by the flow of sell market orders, M_B. With this strategy, the investor buys at a lower price, holds the same quantity of the asset, and saves $A - P > 0$. □

3.2 The Steady state equilibrium

In this section I describe the equilibrium steady state. I consider unspecified equilibrium bid and ask prices B and A, in the price grid $\mathcal{P}$.

Steady state strategy. Investors can trade at price A or price B. At these prices, investors of type *hn* and *lo* want to trade. They play a mixed strategy between limit and market orders, and after trading they become investors of respective types *ho* and *ln*. Yet at these prices, investors of type *ho* or *ln* are better-off not trading. My next proposition specifies the investors' equilibrium strategies.

Proposition 3. *Upon a change in preferences, an investor contacts the market and takes one of the following actions:*

- *If her type is ho, then she cancels any sell limit order she has in the book and stays out of the market thereafter.*
- *If her type is ln, then she cancels any buy limit order she has in the book and stays out of the market thereafter.*
- *If her type is hn, then she places either a buy market order (at price A) with probability m_A or a buy limit order (at price B) with probability $1 - m_A$.*
- *If her type is lo, then she places either a sell market order (at price B) with probability m_B or a sell limit order (at price A) with probability $1 - m_B$.*

Investors of type *ho* or *ln* never have a limit order in the order book. Investors of types *lo* and *hn* either (a) use a one-unit market order and immediately become respective types *ln* and *ho* or (b) post a one-unit limit order in the book. In equilibrium, then, the market depth at prices A and B is equal to the number of investors with types lo and hn, respectively; that is, $D_A = L_{lo}$ and $D_B = L_{hn}$. Moreover, the aggregate state of the limit order book does not change over time; hence D_A and D_B are constant. Proposition 1 implies that there exists an $\alpha \in \mathbb{R}$ such that $L_{lo} = L_{hn} = \alpha$. Thus the free parameter α is determined as the equilibrium market depth.

At each instant, the flows of investors whose private valuation changes - and hence need to buy or sell - are (respectively) equal to ρL_{ln} and ρL_{ho}, that is $\rho\left(\frac{1}{2} - \alpha\right)$ in both cases. It then follows from the strategies employed by investors that the flows of buy and sell market orders are $m_A \rho L_{ln}$ and $m_B \rho L_{ho}$. According to Proposition 2, the equilibrium bid-ask spread must be equal to one tick.

Limit order execution rates. Given the investors' strategies and the market depth α, I can define the *execution rate* of a sell limit order at price A, denoted l_A, as the intensity of the Poisson process that determines the execution time. At each instant, the flow of types ln who become hn and contact the market is equal to $\rho\left(\frac{1}{2} - \alpha\right)$. A fraction m_A sends market orders. As the "random matching" rule applies, a given sell limit order has the same probability of execution as any other order within the mass α. The intensity of execution is therefore written as,

$$l_A = \frac{m_A \rho\left(\frac{1}{2} - \alpha\right)}{\alpha}. \tag{6}$$

The execution rate of a buy limit order, denoted l_B, is similarly defined as

$$l_B = \frac{m_B \rho\left(\frac{1}{2} - \alpha\right)}{\alpha}. \tag{7}$$

In the analysis to follow, I solve for the limit order execution rates l_A and l_B and not for m_A and m_B. This approach is more convenient because l_A and l_B are immediately evident from investors' value functions, as described next, and facilitate derivation of the indifference condition. More precisely, limit orders offer a better execution price than market orders but incur an execution delay. Once the execution rates l_A and l_B and the depth α are determined in equilibrium, one can then calculate m_A and m_B using equations (6) and (7).

Value functions. Equilibrium strategies of the four investor types generate a system of Bellman equations that define the value functions for each type. These value functions depends on the limit order execution rates. The equilibrium values of l_A and l_B are,determined by the indifference conditions between limit and market orders for investors of type lo or hn.

At each instant, an investor of type ho may receive - with Poisson intensity r - the payoff v from the asset. He may (with intensity ρ) also become a lo type and thereby obtain the continuation value V_{lo}. Hence the investor's value function V_{ho} is such that

$$(r+\rho)V_{ho} = rv + \rho V_{lo}. \tag{8}$$

If an investor of type hn has placed a buy limit order at price B then, at each instant, her order may (with Poisson intensity l_B) be executed. In the event that it is executed, the investor becomes an ho type and her continuation value is then $V_{ho} - B$. Alternatively, with intensity ρ she may become an ln type; then her value function V_{hn} is such that

$$(r+\rho+l_B)V_{hn} = \rho V_{ln} + l_B(V_{ho} - B). \tag{9}$$

If a type-hn investor has instead placed a market order, then her continuation value is $V_{ho} - A$. In deciding which order to place, the investor compares the value of a limit order (as given by equation (9)) with the value of the market order net of the payment A. Thus she compares, conditional on all possible next events, the difference in continuation values between the market order net of the payment A and the limit order. Here the possible events are: (i) end of the game, with intensity r; (ii) change in private valuation, with intensity ρ; or (iii) limit order execution, with intensity l_B. All these conditional differences sum to the unconditional difference in continuation value, $V_{ho} - A - V_{hn}$, as follows:

$$(r+\rho+l_B)(V_{ho} - A - V_{hn}) = r(v-A) + \rho(V_{lo} - A - V_{ln}) + l_B[\overbrace{V_{ho} - A - (V_{ho} - B)}^{=-\Delta}]. \tag{10}$$

The investor is indifferent between limit and market orders if and only if $V_{hn} = V_{ho} - A$. Similarly, a type-lo investor is indifferent between limit and market orders if and only if $V_{lo} = V_{ln} + B$. Incorporating these conditions into equation (10) yields

$$(\rho + l_B)\Delta = r(v-A). \tag{11}$$

The right-hand side of equation (11), $r(v-A)$, is the benefit of a market order: conditional on the end of the game, it yields the net benefit of trade $(v - A)$ whereas a limit order does not. The left-hand side, $(\rho + l_B)\Delta$, is the benefit of a limit order: conditional on limit order execution, it saves the investor Δ as compared with a market order. Conditional on a private valuation change, the limit order is simply cancelled; with a market order, in contrast, the asset share must be sold again - and thus incurs the round-trip cost (i.e., the bid-ask spread Δ). In equilibrium, the limit order execution rate l_B adjusts to obtain the indifference condition as follows:

$$l_B = \frac{rv - rA - \rho\Delta}{\Delta} \tag{12}$$

In the same way, we can calculate the value functions of investor types lo and ln, where the equilibrium execution rate is now

$$l_A = \frac{rB - \rho\Delta - rv + \delta}{\Delta} \tag{13}$$

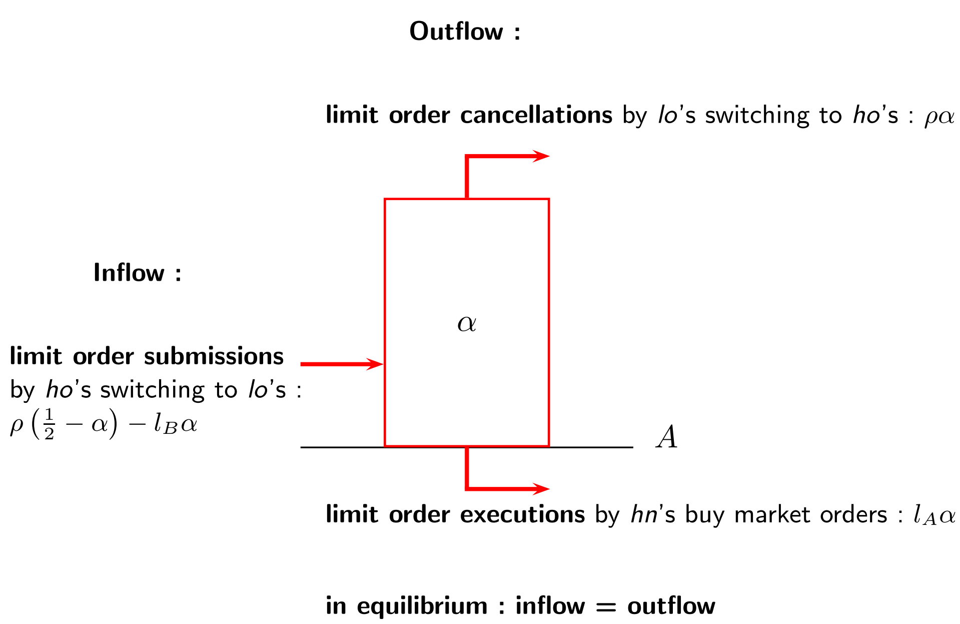

Figure 1: Steady-state dynamics of market depth.

Micro-level dynamics of the limit order book. In the steady state, inflows and outflows of limit orders must - on each side of the order book - exactly compensate each other. Figure 1 shows all order flows on the ask side in the steady-state condition. The inflow formula requires a one-step calculation. The flow of investors who become *lo* and then place sell limit orders is equal to $(1-m_B)\rho\left(\frac{1}{2}-\alpha\right)$. It can be replaced by $\rho\left(\frac{1}{2}-\alpha\right)-l_B\alpha$ because, by equation (7), the flow of sell market orders is equal to the flow of buy limit order executions: $m_B\rho\left(\frac{1}{2}-\alpha\right)=l_B\alpha$. Hence the steady-state condition for the ask side is

$$\rho\left(\frac{1}{2}-\alpha\right)-\rho\alpha-l_A\alpha-l_B\alpha=0. \tag{14}$$

The same reasoning applies on the bid side and yields exactly the same steady-state condition. Therefore, market depth α depends on the limit order execution rate as follows:

$$\alpha=\frac{1}{2}\frac{\rho}{2\rho+l_A+l_B} \tag{15}$$

One can see from equation (15) that α depends on the average limit order execution rate, defined as follows,

$$l^*=\frac{l_A+l_B}{2}=\frac{\delta-(r+2\rho)\Delta}{2\Delta}. \tag{16}$$

Thus, market depth takes the value α^*, which has the following expression,

$$\alpha^*=\frac{1}{2}\frac{\rho\Delta}{\delta-r\Delta} \tag{17}$$

The following proposition gives the conditions under which the previous analysis defines an equilibrium.

Proposition 4. *For each couple of bid and ask prices (A,B) that verifies the conditions*

$$\frac{v}{r}-\frac{\delta}{r}+\frac{\rho}{r}\Delta\leq B<A\leq\frac{v}{r}-\frac{\rho}{r}\Delta, \tag{18}$$

$$A-B=\Delta, \tag{19}$$

there is a unique steady-state equilibrium in which, limit order execution rates, l_A and l_B, and market depth α, are as in equations (12), (13), and (17). Moreover, the equilibrium value *functions are as followed,*

$$\begin{aligned}
V_{ho} &= \frac{1}{r}\frac{1}{2}(rv - \rho\Delta) + \frac{1}{r+2\rho}\frac{1}{2}(rv + \rho(A+B)), \\
V_{ln} &= \frac{1}{r}\frac{1}{2}(rv - \rho\Delta) - \frac{1}{r+2\rho}\frac{1}{2}(rv + \rho(A+B)), \\
V_{hn} &= V_{ho} - A, \\
V_{lo} &= V_{ln} + B.
\end{aligned}$$

4 Equilibrium outcomes

4.1 Limit order execution rates

The equilibrium limit order execution rates are such that investors with types lo and hn are indifferent between getting execution immediacy with a market order or having a delayed execution (for which the loss is measured by the time discount rate r) at a better price with a limit order. On the ask and on the bid side of the market these equilibrium execution rates are respectively equal to l_A and l_B, defined in equations (13) and (12). On average a sell (resp. buy) limit order remains in the book during a time $1/l_A$ (resp. $1/l_B$) before being executed. This is the maximum average time during which an investor with type lo (resp. hn) is willing to wait with a limit order in the order book.

Limit order opportunity cost and execution rates. Depending on the equilibrium prices (A, B), with $A - B = \Delta$, the sell side or the buy side of the market extract more of the trading surplus. Indeed the higher are A and B, the smaller is l_B and the bigger is l_A. An investor with type hn requires a lower execution rate, is willing to wait longer in the book, because his opportunity cost for not trading immediately, $v - rA$, declines. This investor is better-off in an equilibrium with high trading prices. Conversely, an investor with type lo requires a higher execution rate, is not willing to wait longer in the book, because his opportunity cost for not trading immediately, $rB - (v - \delta)$, increases.

Trading surplus extraction, tick size and execution rates. One dimension of the incentive to use a limit order is the opportunity to extract more of the trade surplus compared to the option to use a market order and sell the asset at a lower price (resp. buy at a higher price). This surplus extraction component is measured by the tick size since it captures the price difference between market and limit orders. The bigger is Δ, the lower are l_A and l_B since investors can extract more of the trading surplus by using limit orders and hence are willing to wait longer in the book before being executed (cf Fig. 2). As we will see in the next subsection, the incentive given to investors to extract trading surplus at the expense of execution immediacy is welfare deteriorating.

Private value volatility and execution rates. The frequency ρ, at which the preference for the asset of an investor switches from high to low or low to high, has a negative effect on the equilibrium execution rates (cf Fig. 2). When ρ is high, investors anticipate that, if they trade immediately after a change in type, they may trade again soon after a switch back of their type. As a consequence, the incentive for trading is less. Investors suffer less from waiting with a suboptimal type, lo or hn, with a limit order in the book.

The asset holding cost δ, that low type investors suffer from, has a positive impact on l_A. This effect goes through the opportunity cost channel. A higher δ implies a higher opportunity cost for lo type investors who use a limit order.

Average execution rate. The formula for the average execution rate l^* (see equation (16)) allows for easily assessing the effects of the model parameters on the execution rates. They are as follows

$$\frac{\partial l^*}{\partial \Delta} < 0, \ \frac{\partial l^*}{\partial \rho} < 0, \ \frac{\partial l^*}{\partial \delta} > 0.$$

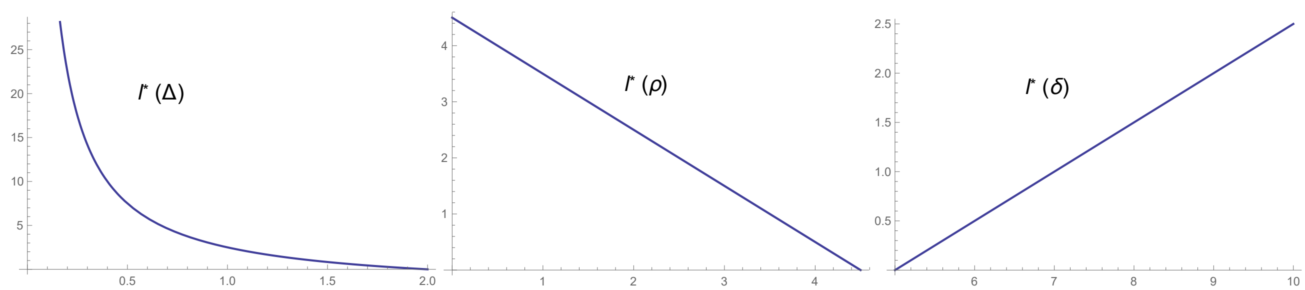

Figure 2: Execution rate l^* in function of (i) $\Delta \in [0, \delta/(r+2\rho)]$ ($\rho = 2$ $\delta = 10$), (ii) $\rho \in [0, (\delta - r\Delta)/2\Delta]$ ($\Delta = 1$, $\delta = 10$) and (iii) $\delta \in [(r+2\rho)\Delta, 2(r+2\rho)\Delta]$ ($\Delta = 1$, $\rho = 2$) , ($r = 1$).

l^* is also the execution rate in the particular case where bid and ask prices are $B = v - \frac{\delta}{2r} - \frac{\Delta}{2}$, and $A = v - \frac{\delta}{2r} + \frac{\Delta}{2}$. In this particular case, the equilibrium quotes are symmetric with respect to the "average" asset value, $v - \delta/2r$. It corresponds to the value of owning the asset for the average investor, that is an investor who would constantly pay a cost $\delta/2$ by asset share owned.

4.2 Market depth

On both bid and ask sides of the market, the market depth (i.e. the number of limit orders submitted) is equal, at each point in time, to α^*, and has the following properties

$$\frac{\partial \alpha^*}{\partial \Delta} > 0, \; \frac{\partial \alpha^*}{\partial \rho} > 0, \; \frac{\partial \alpha^*}{\partial \delta} < 0.$$

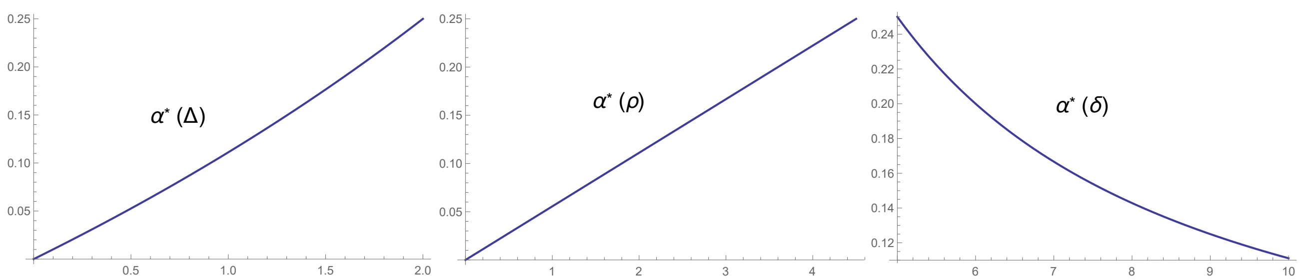

Figure 3: Market depth α^* in function of (i) $\Delta \in [0, \delta/(r+2\rho)]$ ($\rho = 2 \; \delta = 10$), (ii) $\rho \in [0, (\delta - r\Delta)/2\Delta]$ ($\Delta = 1, \; \delta = 10$) and (iii) $\delta \in [(r+2\rho)\Delta, 2(r+2\rho)\Delta]$ ($\Delta = 1, \; \rho = 2$) , ($r = 1$).

The tick-size Δ has a positive effect on the market depth since a tick-size increase generates a larger trading surplus that one can extract with limit order (see Fig. 3).

The parameter ρ has a positive effect on the market depth since a higher ρ implies that, at each time t, there is an increasing fraction of investors whose type have become suboptimal and thus have trading need, lo and hn, which has a positive effect on the number of limit order submitted. This effect is amplified by a lower equilibrium limit order execution rate.

δ has a negative effect on the market depth. Through the limit order opportunity cost channel, a higher δ imposes a higher execution rate which implies a lower market depth.

4.3 Trading intensity/volume

On the ask and the bid side of the market, the trading intensities, are respectively equal to $l_A \alpha^*$ and $l_B \alpha^*$. Hence the overall trading intensity is

$$2l^* \times \alpha^* = \frac{\delta - (r+2\rho)\Delta}{\Delta}\alpha^* = \frac{\rho}{2}\frac{\delta - (r+2\rho)\Delta}{\delta - r\Delta}$$

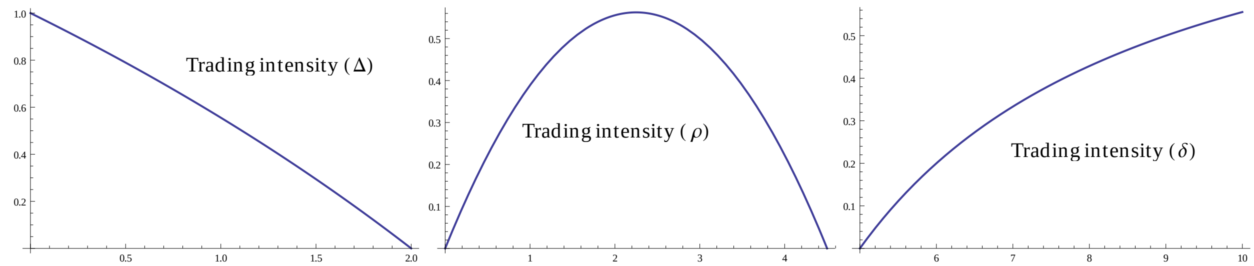

Figure 4: Trading intensity in function of (i) $\Delta \in [0, \delta/(r+2\rho)]$ ($\rho = 2\ \delta = 10$), (ii) $\rho \in [0, (\delta - r\Delta)/2\Delta]$ ($\Delta = 1,\ \delta = 10$) and (iii) $\delta \in [(r+2\rho)\Delta, 2(r+2\rho)\Delta]$ ($\Delta = 1,\ \rho = 2$) , ($r = 1$).

The overall trading volume can be calculated as the integral of the trading intensity over the all game period, $[0, +\infty)$ multiplied by the average trading horizon, $1/r$.

The tick-size Δ has a negative effect on the trading intensity since, everything else equal, an increase of this parameter increases the trading surplus that one can extract with limit order. Thus the equilibrium execution rate declines. The number of limit order α^* increases. The overall effect on the trading intensity is negative (see Fig. 4).

The effect of the parameter ρ is not monotonic. On the one hand a higher ρ implies that, at each time t, there is an increasing fraction of investors whose type have become suboptimal and thus have trading need, lo and hn, which has a positive effect. On the other hand, a higher ρ reduces the intensity of this trading need since investors anticipate that they may more likely switch back to an optimal type. As a consequence the equilibrium execution rates decline. The overall effect is positive for low ρ's and negative for high ρ's.

δ has a positive effect on the trading intensity. Through the limit order opportunity cost channel it imposes a higher equilibrium execution rate. This effect is mitigated by a lower implied market depth but still remain positive.

5 Welfare analysis

Proposition 5. *For any steady-state equilibrium with bid and ask prices* (A, B)*, the level of welfare* W *is the same and equal to*

$$W = \frac{1}{r}\frac{1}{2}(rv - \rho\Delta) - \alpha^*\Delta = \frac{v}{2} - \alpha^*\frac{\delta}{r}. \tag{20}$$

The welfare is impacted negatively by ρ *and the tick-size* Δ*, and positively by* δ*,*

$$\frac{\partial W}{\partial \Delta} < 0, \ \frac{\partial W}{\partial \rho} < 0, \ \frac{\partial W}{\partial \delta} > 0.$$

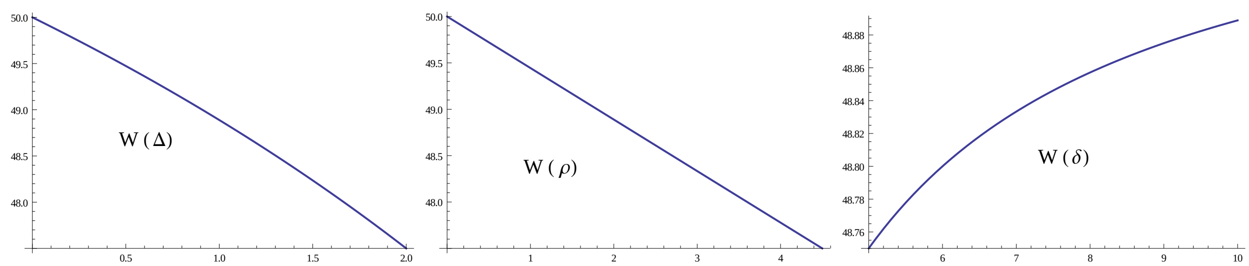

Figure 5: Welfare W in function of (i) $\Delta \in [0, \delta/(r + 2\rho)]$ $(\rho = 2 \ \delta = 10)$, (ii) $\rho \in [0, (\delta - r\Delta)/2\Delta]$ $(\Delta = 1, \ \delta = 10)$ and (iii) $\delta \in [(r+2\rho)\Delta, 2(r+2\rho)\Delta]$ $(\Delta = 1, \ \rho = 2)$, $(v = 100, \ r = 1)$.

The maximum level of welfare than can be reached is equal to $\frac{v}{2}$. It is obtained when all the asset supply $\frac{1}{2}$ is owned by high type investors whose population size is $\frac{1}{2}$ as well. Reaching this optimum requires that when *hn* and *lo* investors come to the market, after they changed of type, they can trade immediately at one price that is the same for buyers and sellers.

The role of the tick size. In a steady-state equilibrium this optimum can be reached if the tick size is nil, $\Delta = 0$, as we can see in the formula for the welfare (cf. equation 20). The tick size is the friction that prevents from reaching the optimum. Because there is a difference in the execution prices for limit and market orders, investors have an incentive to send limit orders and to wait for execution whereas it would be socially optimal that these orders get immediately executed. The corresponding loss is captured by the term $-\alpha^*\frac{\delta}{r}$. It shows that the presence of liquidity supply, α^* is suboptimal.

When $\Delta = 0$, the bid and the ask prices are infinitely close. Then the price improvement of submitting limit orders is nil and the execution intensities l_A and l_B must be infinite to incentivize limit order submission. Because of these infinite execution rates, limit orders are instantaneously executed and the limit order book is always empty. We can view this equilibrium as a situation where investors coordinate to trade with each other at a single price $P = A = B$ and where there is no difference between limit and market orders.

Effects of the preference volatility components. The idiosyncratic preference switching frequency ρ at which has a negative effect on the welfare (cf Fig. 5) through its increasing effect on α^*. The asset holding cost δ, for low type investors, has a positive effect on the welfare.

It is interesting to compare these effects with the benchmark case where there is no trading. In this case the level of welfare is given by the value function of investors with type lo and ho, and the initial fractions of these investors type. These value function are as followed,

$$V_{lo} = v - \frac{\delta}{r}\frac{r+\rho}{r+2\rho}, \quad V_{ho} = v - \frac{\delta}{r}\frac{\rho}{r+2\rho}.$$

Depending on the initial level of populations, the effect of ρ can be positive or negative since it has a positive effect for lo type (they switch to a high private value faster) and a negative effect for ho type. For instance, in the steady-state case, the initial fraction of each investor would be 1/4 and the welfare would be equal to $\frac{v}{2} - \frac{\delta}{4r}$ and the parameter ρ has no effect. In comparison, with trading, welfare becomes sensitive to parameter ρ.

The effect of δ is very clear in the situation without trading since, everything else equal, increasing this cost induces an actual or an expected utility loss for all investors who own the asset. Thus it noticeable that the effect of this cost is reversed when investors can trade. It generates an opportunity cost for using limit orders which accelerate trading and make the asset allocation across investor more optimal.

Model outcomes and proxies for investor's welfare. To empirically investigate the source of welfare variations for investors in a given financial market, we need observable proxies for welfare. Usually liquidity measures are thought as positively related to investor's welfare, since higher market liquidity leads to a lower implicit trading cost. My model provides counter intuitive results in this respect.

The previous results for welfare and market depth shows that any variation of parameters Δ, ρ or δ has opposite effects these model outcomes. Market depth measures the level of liquidity supply in the market but in negatively related to investors welfare, at least in this model. Trading intensity which is also a usual liquidity measures co-varies much better with the welfare (cf. Fig 6) except for variations of ρ when it has low values. The only model outcomes that always positively co-varies with investor's welfare for any parameter's variation is the limit order execution rate.

	Limit order execution rate (l^*)	Market Depth (α^*)	Trading Intensity	Welfare
$\frac{\partial}{\partial \Delta}$	$-$	$+$	$-$	$-$
$\frac{\partial}{\partial \rho}$	$-$	$+$	$+/-$	$-$
$\frac{\partial}{\partial \delta}$	$+$	$-$	$+$	$+$

Figure 6: Signs of first order partial derivatives of model's outcomes with respect to model's parameters Δ, ρ and δ.

	Limit order execution rate (l^*)	Market Depth (α^*)	Trading Intensity	Welfare
$\frac{\partial}{\partial \Delta \partial \rho}$	0	$+$	$-$	$-$
$\frac{\partial}{\partial \Delta \partial \delta}$	$-$	$-$	$+$	$+$
$\frac{\partial}{\partial \rho \partial \delta}$	0	$-$	$+$	$+$

Figure 7: Signs of second order cross partial derivatives of model's outcomes with respect to model's parameters Δ, ρ and δ.

One could also want to investigate the effect on liquidity and welfare of the change of one model parameter across different markets that could be sorted with respect to a second parameter. For instance one could look at the effect of decimalization, a reduction of Δ, on the cross section of security markets sorted with respect to the holding cost δ (which would imply using a proxy for such a cost though). To implement such an empirical analysis and draw conclusion on the cross sectional effect of such a shock on welfare, one would need a welfare proxy for which the second order cross partial derivative, with respect to the “shocked” parameter and the parameter used to sort the markets cross section, has at least the same sign as the corresponding derivative for the welfare. In our setup, trading intensity would the best proxy (cf. Fig 7).

6 Conclusion

Market depth should measure implicit trading costs for liquidity consumers. However, in limit order markets, market depth does not reflect the execution quality of limit orders which are used by traders who decide to supply liquidity. Hence, market depth, as a liquidity measure, may not be sufficient to infer investors’ welfare. In this paper, I show with a model that market depth can be negatively related to investors welfare because high a market depth reflects a low execution rate of limit orders and a relatively low speed of gains from trade realizations. In this context, considering new liquidity measures as limit order execution rates, or trading volume, may help to better assess the link between market conditions and investors’ welfare.

A Proofs

A.1 Proof of proposition 1

The steady state is defined by the system

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{ho} \\ L_{hn} \\ L_{lo} \\ L_{ln} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

First we can check that $L_{ho} = sL,\ L_{hn} = 0,\ L_{lo} = 0,\ L_{ln} = \frac{1}{2}$ is a particular solution of this system. The general space of solutions of this system is equal to

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \ker \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + Vect \left[\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right]$$

A.2 Proof of proposition 4

type ho**.** An investor of type ho keeps its asset until he/she switches to the lo type, with intensity ρ, or until the asset pays-off, with intensity r,

$$(r + \rho)V_{ho} = \rho V_{lo} + rv$$

type ln**.** An investor of type ln does not sen any order until he/she switches to the hn type.

$$(r + \rho)V_{ln} = \rho V_{hn}$$

type hn**.** An investor of type hn send a buy market with probability m_A or a buy limit order with probability $1 - m_A$. At time t, the outflow of the bid side due to market order is $m_B \rho L_{ho}.dt$, then the probability for a limit order to be executed at t is $l_B.dt$ with

$$l_B = \frac{m_B \rho L_{ho}}{L_{hn}}$$

When such an investor sends a market order or a limit order, they obtain respectively, in term of value function

$$V_{hn} = V_{ho} - A$$

or

$$(r + \rho + l_B)V_{hn} = \rho V_{ln} + l_B(V_{ho} - B)$$

In both case V_{hn} must coincide, hence

$$(r + \rho + l_B)(V_{ho} - A) = \rho V_{ln} + l_B(V_{ho} - B)$$

type lo**.** An investor of type hn send a sell market with probability m_B or a sell limit order with probability $1 - m_B$. For the same reason as for type hn, at time t, the outflow of the ask side due to market order is $m_A \rho L_{ln}.dt$, then the probability for a limit order to be executed at t is $l_A.dt$ with

$$l_A = \frac{m_A \rho L_{ln}}{L_{lo}}$$

As in the previous case, the value function verifies

$$V_{lo} = V_{ln} + B$$

and

$$(r + \rho + l_A)V_{lo} = rv - \delta + \rho V_{ho} + l_A(V_{ln} + A)$$

leading to

$$(r + \rho + l_A)(V_{ln} + B) = v - \delta + \rho V_{ho} + l_A(V_{ln} + A)$$

First, by replacing V_{hn} by $V_{ho} - A$ and V_{lo} by $V_{ln} + B$ this is easy to obtain that

$$(r + \rho)V_{ho} - \rho V_{ln} = v + \rho B$$
$$(r + \rho)V_{ln} - \rho V_{ho} = -\rho A$$

and then to get the expression of V_{ho} and V_{ln}.
Replacing V_{ln} by $V_{lo} - B$ and V_{ho} by $V_{hn} + A$ in the equation of indifference between market and

limit orders we obtain

$$(r+\rho+l_B)(V_{ho}-A)=\rho(V_{lo}-B)+l_B(V_{ho}-B)$$
$$(r+\rho+l_A)(V_{ln}+B)=rv-\delta+\rho(V_{hn}+A)+l_A(V_{ln}+A)$$

which gives

$$v+\rho B-(r+\rho)A=l_B(A-B)$$
$$-\rho A+(r+\rho)B-(rv-\delta)=l_A(A-B)$$

l_A and l_B must be positive numbers. Hence $rv+\rho B-(r+\rho)A=rv-\rho\Delta-rA>0$ and $-\rho A+(r+\rho)B-(rv-\delta)=rB-(rv-\delta)-\rho\Delta>0$.

A.3 Proof of proposition 3

In this proof, I refer to value functions introduced in the proof of proposition 4. I show that it is not possible to find a profitable one-shot deviation from the strategy exposed in 3. This is a necessary and sufficient condition as shown in Lemma A.1 of the Online Appendix.

type hn**.** A type hn has only one way to deviate which is to stay out of the market and not sending a buy market or limit order. Actually we could consider a change of the mixed strategy parameter between limit and market orders but given that the investor is infinitesimal this deviation does not change the value of these two actions. Then he/she is indifferent between all mixed strategies which makes this deviation not profitable.

The value of not trading when contacting the market is

$$V_{hn-out}=(1-r.dt)[(1-\rho.dt)V_{hn-out}+\rho.dtV_{ln}$$
$$\Longleftrightarrow(r+\rho)V_{hn-out}=\rho V_{ln}\leq\rho V_{hn}$$

if $V_{ln}\leq V_{hn}$ which is true if and only if $V_{ln}\geq 0$ because $(r+\rho)V_{ln}=\rho V_{hn}$. In this case this deviation is not profitable. It is easy to verify that $V_{ln}\geq 0$ iff

$$A\leq\frac{1+\rho B}{r+\rho}$$

which is verified as soon as $l_B>0$.

type lo**.** As in the previous case the only deviation we have to consider is the case where a type lo in contact with the market decides to keep the asset. In this case the value function is

$$(r+\rho)V_{lo-out} = rv - \delta + \rho V_{ho}$$

As we know that $(r+\rho)(V_{lo} - B) = \rho(V_{ho} - A)$ we get

$$(r+\rho)V_{lo-out} = rv - \delta + \rho(A - B) - rB + (r+\rho)V_{lo}$$

Then the deviation is not profitable iff

$$rv - \delta + \rho(A - B) - rB \leq 0 \Longleftrightarrow B \geq \frac{1 - \delta + \rho A}{r + \rho}$$

which is verified as soon as $l_A > 0$.

type ho**.** A type ho can deviate in two different ways:

- by sending a sell market order. The value function associated is then $V = V_{hn} + B = V_{ho} - A + B < V_{ho}$. This is not profitable.
- by sending a sell limit order. In this case the value function is given by

$$(r+\rho+l_A)V_{ho-A} = v + \rho V_{lo} + l_A V_{ho} + l_A(V_{hn-out} - V_{hn}) < (r+\rho+l_A)V_{ho}$$

  This deviation is not profitable.

type ln**.** A type ln can deviate in two different ways:

- by sending a buy market order. The value function associated is then $V = V_{lo} - A = V_{ln} + B - A < V_{ln}$. This is not profitable.
- by sending a buy limit order. In this case the value function is given by

$$(r+\rho+l_B)V_{ln-B} = \rho V_{hn} + (\lambda + l_B)V_{ln} + l_B(V_{lo-out} - V_{lo}) < (r+\rho+l_B)V_{ln}$$

  This deviation is not profitable.

The two steady state equations can be rewritten as

$$\rho L_{hn} + l_B L_{hn} + l_A L_{lo} = \rho L_{ln}$$

$$\rho L_{lo} + l_A L_{lo} + l_B L_{hn} = \rho L_{ho}$$

replacing by the possible value of these masses at equilibrium we obtain the equations in function of α

$$\rho\alpha + l_B\alpha + l_A\alpha = \rho\left(\frac{1}{2} - \alpha\right)$$

$$\rho\alpha + l_A\alpha + l_B\alpha = \rho\left(\frac{1}{2} - \alpha\right)$$

that both give the same result for α_{eq}

$$\alpha = \frac{\frac{\rho}{2}}{2\rho + l_A + l_B}$$

To be sure that the equilibrium exists we need to check that $0 \leq m_A \leq 1, \quad 0 \leq m_B \leq 1$

$$m_A = \frac{\alpha}{\rho(\frac{1}{2} - \alpha)} l_A = \frac{l_A}{\rho + l_A + l_B} < 1$$

and for the same reason

$$m_B = \frac{l_B}{\rho + l_A + l_B} < 1$$

A.4 Proof of proposition 5

This proof is straightforward,

$$W = \left(\frac{1}{2} - \alpha^*\right) \times (V_{ho} + V_{ln}) + \alpha^* \times (V_{lo} + V_{hn}) = \frac{1}{r}\frac{1}{2}(rv - \rho\Delta) - \alpha^*\Delta = \frac{v}{2r} - \alpha^*\frac{\delta}{r}.$$

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