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Large Portfolio Risk Management and Optimal Portfolio Allocation with Dynamic Elliptical Copulas

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Abstract

Previous research focuses on the importance of modeling the multivariate distribution for optimal portfolio allocation and active risk management. However, available dynamic models are not easily applied for large-dimensional problems due to the curse of dimensionality. In this paper, we extend the framework of the Dynamic Conditional Correlation/Equicorrelation and an extreme value approach into a series of Dynamic Conditional Elliptical Copulas. We investigate risk measures like Value at Risk (VaR) and Expected Shortfall (ES) for passive portfolios and dynamic optimal portfolios through Mean-Variance and ES criteria for a sample of US stocks over a period of 10 years. Our results suggest that (1) Modeling the marginal distribution is important for the dynamic high-dimensional multivariate models. (2) Neglecting the dynamic dependence in the copula causes over-aggressive risk management. (3) The DCC/DECO Gaussian copula and t-copula work very well for both VaR and ES. (4) Grouped t-copula and t-copula with dynamic degrees of freedom further match the fat tail. (5) Correctly modeling dependence structure makes an improvement in portfolio optimization against the tail risk. (6) Models driven by multivariate t innovations with exogenously given degrees of freedom provide a flexible and applicable alternative for optimal portfolio risk management.

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1 Introduction

Modeling dynamic multivariate distribution is of crucial importance in finance. The cross-sectional information in dynamic multivariate models is extremely useful for active risk management such as computing the marginal risk contributions of each position, evaluating the effects of hedges, and constructing an optimal portfolio. The copula theory is a fundamental tool for modeling multivariate distributions. It allows the definition of the joint distribution through the marginal distributions and the dependence between the variables. Correlation, which usually refers to linear correlation, depends on both the marginal distributions and the copula, and is not a robust measure, as a single observation can have an arbitrarily strong impact. The copula provides a robust method of consistent estimation for dependence, and is much more flexible.

The copula theory has been extended to the conditional case, allowing the use of copulas to model dynamic structures, as in Dias and Embrechts (2004), Patton (2004, 2006a&b) and Jondeau and Rockinger (2003, 2006). Compared with traditional methods of Value at Risk (VaR) estimation, conditional copula theory can be a very powerful tool in estimating the VaR, as shown by Fantazzini (2008) and Palaro and Hotta (2006). However, modelling large-dimensional distributions is not an easy task and only a few models are potentially useful for constructing flexible distribution models in large dimensions. Lee and Long (2009) decompose the dependence into linear and nonlinear component, and use copula-based models for the nonlinear component. Nevertheless, their approach is computationally infeasible in high dimensions as discussed by Oh and Patton (2016). Latent factor copula models, see Creal and Tsay (2015) and Oh and Patton (2017) for example, are particularly attractive for relatively large dimensional applications to the factor structure, but factor copula models do not generally have a closed-form likelihood, making maximum likelihood estimation difficult. Vine copulas are constructed by sequentially applying bivariate copulas to build up a larger-dimension copula, see, e.g., Almeida, Czado, and Manner (2012) and Brechmann and Czado (2013). However, as shown by Acar, Genest, and Neslehova (2012), vine copulas are almost invariably based on an assumption that is hard to interpret and to test. Extreme-value copulas such as Archimedean copulas provide appropriate models for the dependence structure between rare events, e.g., Patton (2004), but usually have one or two free parameters regardless of number of asset, which is clearly very restrictive in large dimensions.

Multivariate GARCH techniques applied in the dynamic correlation multivariate models are called Dynamic Conditional Correlation Multivariate GARCH (DCC). It has proven to be difficult to estimate multivariate GARCH models for a large number of underlying securities owning to the curse of dimensionality; the estimates are usually seriously biased. However, recently several estimation methods been developed to treat this large-dimensional problem: the MacGyver method (Engle (2009)), Dynamic Conditional Equicorrelation (DECO) (Engle and Kelly (2012)), and the Composite Likelihood method (Engle, Shephard, and Sheppard (2008)). The DCC or DECO approach puts a dynamic multivariate distribution on top of the dynamic marginal distributions and therefore can be viewed as a simple kind of dynamic copula approach without modeling marginal distributions specifically. Christoffersen et al. (2012) allow the DCC structure in a skew t copula for their dependence study of 33 developed and emerging equity market indices, and Christoffersen et al. (2017) used the same model to study 233 equity returns and credit default swap spreads.

In the light of the recent development of multivariate GARCH techniques for a large number of underlying securities, in this paper, we extend the DCC/DECO framework to more general Dynamic Conditional Elliptical Copulas, the Gaussian- and t-copula. Two good candidates that are especially tractable for large dimensions. The class of elliptical distributions provides very useful examples of multivariate distributions because they share many of the tractable properties of the multivariate normal distribution. Furthermore, they allow modeling of multivariate extreme events and forms of non-normal dependencies. They play an important role in finance. Elliptical copulas are simply the copulas of elliptical distributions. We limit the choices of copula within the class of elliptical copulas because (1) elliptical copulas allow us to specify a variance-covariance structure which in some senses provides the linear dependence between the random variables; (2) except for the normal copula which gives zero tail dependence, elliptical copulas also allow for non-zero tail dependence; (3) in addition, elliptical copulas provide the flexibility of simple simulation procedures. Therefore the class of elliptical copulas is a good candidate for the large-dimensional problem.

In this study, we adopt a semi-parametric form of the marginal distributions and new estimation methods for multivariate GARCH models. We propose a series of dynamic copulas to examine VaR and ES accordingly not only for passive large portfolios but also for dynamic optimal large portfolios by both Mean-Variance and ES criteria. Assuming equal correlation or dependence across assets is very convenient and widely used by the financial industry: for instance, in the CDO market, the standard one-factor Gaussian copula also assumes equal dependence. Elton, Gruber, and Green (2007) find that, despite all the advances in modeling, the assumption that all correlations are identical to each other still serves about as well as any other forecasting technique for portfolio optimization. Therefore we further explore if equal correlation or dependence performs about as well as full dependence structure for active risk management.

In our empirical study, we consider a large sample of US stocks during the period 1995-2005, which consists of 89 North American investment grade companies included in the CDX.NA.IG, a credit default swap index. This sample period which includes an apparent change of dependence is perfectly suitable for highlighting the importance of the copula-based dependence approach compared with the traditional correlation analysis. We construct equally-weighted portfolios and value-weighted portfolios which are updated annually. The out-of-sample performances in terms of VaR and ES show that (1) Modeling the marginal distributions well is very important. In DCC or DECO, simulation directly from multi-normal distribution or multi-t distribution does not induce enough tail fatness. The assumption that the portfolio returns are from univariate conditional t-distribution, with portfolio variance computed from conditional covariance matrix from DCC or DECO, indeed recovers tail fatness further, but at the sacrifice of the upper quantiles. (2) Neglecting dynamic dependence alone in the copula can cause over-aggressive risk management. The proportion of excessive losses can be seriously underestimated when real dependence is moving up and down, for example, in the post high-tech bubble period. (3) DECO and DCC Gaussian copulas and t-copulas work very well in both VaR and ES. The DCC copula does a better job, especially for the value-weighted portfolio, by taking care of the widely dispersed dependence structures. For the equally-weighted portfolio, the DECO copula still performs about as well as the DCC copula. (4) Grouped t-copula and full dynamic t-copula, in which the degree of freedom is also modeled dynamically, match the fat tail even further, with an improvement especially at the lower quantiles.

The optimal portfolios are constructed empirically by the Markowitz (1952) portfolio selection method. When asset returns are far from being normally distributed, the traditional mean-variance framework may produce misleading results. We therefore minimize the portfolio's ES also by the scenario-based copula models as Rockafellar and Uryasev (2000) and Andersson et al. (2000). We find that (1) Even though the assumption of all correlations identical to each other is convenient, correctly modeling dependence dispersion can still make a marginal improvement in portfolio optimization, which should not be neglected. The optimal portfolio by ES does a better job against the tail risk compared to the optimal portfolio by mean-variance in both DECO and DCC copulas. So the Markowitz (1952) portfolio selection is not efficient if we consider the tail risk. (2) Because optimizers actually act as statistical error maximizers, the ex ante VaR and ES by simulation in the framework of the copula underestimate the loss even under short-selling restrictions. Instead, models driven by multivariate t innovations with exogenously given degrees of freedom actually works very well for optimal portfolio risk management.

The rest of the paper proceeds as follows. In Section 2, we provide a brief review of elliptical copulas and several scalar measures of dependence, explain how to model the marginal dynamics, and introduce a series of dynamic copulas by extending the multivariate GARCH models (DCC/DECO) into the framework of the Gaussian copula and t-copula. Section 3 shows the empirical applications and the backtesting results. Finally, conclusions are drawn and future works discussed in Section 4.

2 Dynamic Elliptical Copula Models

Patton (2006b) defined a "conditional copula" as an n-dimensional multivariate distribution of variables $X_t = [X_{1t}, X_{2t}, ..., X_{nt}]'$ conditional on some information set $\mathcal{F}_{t-1} = \sigma(X_{t-j}; j \geq 1)$ for $i = 1, 2, ..., n$ and $t \in \{1, ..., T\}$. We adopt the usual convention of denoting random variables in upper case, and realizations of random variables in lower case. Based on this definition, we can extend Sklar's theorem to the time series case:

$$F(x|\mathcal{F}_{t-1}) = C(F_1(x_1|\mathcal{F}_{t-1}), F_2(x_2|\mathcal{F}_{t-1}), ..., F_n(x_n|\mathcal{F}_{t-1})|\mathcal{F}_{t-1}), \forall x \in \mathbb{R}^n$$

where $X_t|\mathcal{F}_{t-1} \sim F(\cdot|\mathcal{F}_{t-1})$, $X_{it}|\mathcal{F}_{t-1} \sim F_i(\cdot|\mathcal{F}_{t-1})$ which is independent and identically distributed with mean zero and variance one by construction, and C is the conditional copula of X_t given $\mathcal{F}_{t-1}$. The Sklar's theorem for conditional distributions implies that the conditioning set, $\mathcal{F}_{t-1}$, must be the same for all marginal distributions and the copula. It is often the case in financial applications, however, that some of the information contained in $\mathcal{F}_{t-1}$ is not relevant for all variables. We might define $\mathcal{F}_{i,t-1}$ as the smallest subset of $\mathcal{F}_{t-1}$ such that $X_{it}|\mathcal{F}_{i,t-1} \overset{D}{=} X_{it}|\mathcal{F}_{t-1}$. With this it is possible to construct each marginal distribution model using only $\mathcal{F}_{i,t-1}$, which will probably differ across margins, and then use $\mathcal{F}_{t-1}$ for the copula, to obtain a valid conditional joint distribution.

We present two copulas belonging to the elliptical family, which will be used later in the empirical applications, namely the Gaussian copula and the t-copula. For the Gaussian copula, its dependence features are characterized by the basic scalar dependence parameter for two variables X and Y:

$$\rho_{GR} = Corr[\Phi^{-1}(F(x)), \Phi^{-1}(F(y))] = Corr[\Phi^{-1}(u), \Phi^{-1}(v))$$

where $u = F(x)$ and $v = F(y)$. ρ_{GR} is the correlation between the variables $\Phi^{-1}(u)$, and $\Phi^{-1}(v)$, which are both marginally standard Gaussian and admit the same copula as X, Y. As other rank correlations such as the so-called Spearman's rho and Kendall's Tau, it can detect nonlinear association that Pearson's linear correlation cannot see. It coincides with the linear correlation if the basic variables are jointly Gaussian. As for the t-copula, the scalar measure of dependence is $\rho_{TR} = Corr[t_v^{-1}(u), t_v^{-1}(v))$. These basic scalar dependence measures are usefull to interpret the parameters or to construct dependent update forms for parametric dynamic copula families.

The choice of the best distribution for the margins is also crucial. Misspecification of marginals can lead to dangerous biases in dependence measure estimation, see, for instance, Fantazzini (2009). Hence, we need a procedure that avoids marginal model risk as much as possible and also does not mask the dependence structure too much. The proposal is known as the canonical maximum likelihood method (CML). The CML estimation procedure consists of two stages. In the first one we fit univariate $ARMA(p_1, q_1) - GARCH\ (p_2, q_2)$ models (for other candidate $GARCH$, please see Bollerslev (2008)) to each of the marginal series X_{it} with innovations ε_{it} assumed to come from $iid(0, 1)$. We can assume ε_{it} comes from a t distribution, Hansen's skewed t distribution or other distributions with heavy-tailedness as in Jondeau and Rockinger (2006), and estimate univariate $ARMA - GARCH$ models by Maximum Likelihood. However fitting a parametric distribution to data sometimes results in a model that agrees well with the data in high-density regions, but poorly in areas of low density. Instead, we do not specify the marginal distribution but adopt a semi-parametric form of the marginal distributions. We estimate the $ARMA - GARCH$ model directly by Quasi-Maximum Likelihood by assuming $\varepsilon_{it} \sim iid(0,1)$, and select the best model by some automatic model selection criteria. Given the standardized $i.i.d$ residuals from the previous step, we estimate the empirical cumulative distribution function (cdf) of each time series data, smoothe the cdf estimates with a Gaussian kernel, and then fit the amount by which those extreme residuals in each tail fall beyond a threshold, e.g. 10% to a parametric GP by maximum likelihood. This approach is often referred to as the distribution of exceedances or peaks-over-threshold method (see, for instance, McNeil (1999), McNeil and Frey (2000) or Nystrom and Skoglund (2002a&b)).

In the second step a parametric family of copulas is fitted. The parameters of the copula are estimated as:

$$\widehat{\theta}_c = \arg\max \sum_{t=1}^{T} \log[c(F_1(x_{1t}|\mathcal{F}_{1,t-1}), F_2(x_{2t}|\mathcal{F}_{2,t-1}), ..., F_n(x_{nt}|\mathcal{F}_{n,t-1})|\mathcal{F}_{t-1};\theta_c)]$$

where c is the conditional copula density function of X_t given $\mathcal{F}_{t-1}$, and we assume ε_{it} or equivalent in general, $X_{it}|\mathcal{F}_{i,t-1} \overset{D}{=} X_{it}|\mathcal{F}_{t-1} \sim iid(0,1)$. In Genest, Ghoudi and Rivest (1995), and Chen and Fan (2006), it is shown that the estimator θ_c for the dependence parameter resulting from CML is consistent and has asymptotically normal distribution under regularity conditions similar to those of maximum likelihood theory. Moreover, the asymptotic variance-covariance matrix of $T^{1/2}\theta_c$ is $B^{-1}\Sigma B^{-1}$, where B is the information matrix associated with the copula and Σ is the variance-covariance matrix of the n-dimensional random vector. The estimation of the variance-covariance matrix is given in Section 3-4 of Genest, Ghoudi and Rivest (1995).

In the next section, we extend the multivariate GARCH techniques (DCC/DECO) into elliptical copulas to investigate the conditional dynamic dependence.

2.1 Dynamic Gaussian Copula

The Gaussian copula is the copula of the multivariate normal distribution. In fact, the random vector $X_t|\mathcal{F}_{t-1} = (\varepsilon_{1t}, ..., \varepsilon_{nt})$ is multivariate normal if the univariate margins $F_1, ..., F_n$ are Gaussians. Let R_t be a symmetric, positive definite matrix with diagonal of R_t equals to 1, and Φ_{R_t} standardized multivariate normal distribution with correlation matrix R_t at time t. The dynamic multivariate Gaussian copula is then defined as follows:

$$C(\eta_1, \eta_2, ..., \eta_n; R_t) = \Phi_{R_t}(\Phi^{-1}(\eta_1), \Phi^{-1}(\eta_2), ..., \Phi^{-1}(\eta_n))$$

where $\eta_i = F_i(\varepsilon_i)$ for $i = 1, 2, ..., n$. $\varepsilon_{it} \sim iid(0,1)$ innovations from the marginal dynamics introduced in the previous section. R_t can be assumed to be constant or a dynamic process through time.

2.1.1 DCC Gaussian Copula

Engle (2002) proposes a class of models - the Dynamic Conditional Correlation (DCC) - that both preserves the ease of estimation of Bollerslev (1990)'s constant correlation model and allows the correlations to change over time. These kinds of dynamic processes can also be extended into Gaussian copulas. The simplest rank correlation dynamics we consider empirically is the symmetric scalar model where the entire rank correlation matrix is driven by two parameters.

$$Q_t = (1 - \alpha - \beta)\overline{Q} + \alpha(\varepsilon^*_{t-1}\varepsilon^{*\prime}_{t-1}) + \beta Q_{t-1}$$

where $\alpha \geq 0, \beta \geq 0, \alpha + \beta < 1$, $\varepsilon^*_i = \Phi^{-1}(\eta_i = F_i(\varepsilon_i))$, $Q_t = |q_{ij,t}|$ is the auxiliary matrix driving the rank correlation dynamics, the nuisance parameters $\overline{Q} = E\left[\varepsilon^*_t \varepsilon^{*\prime}_t\right]$ with sample analog $T^{-1}\sum_{t=1}^{T} \varepsilon^*_t \varepsilon^{*\prime}_t$, so that R_t is a matrix of rank correlations $\rho_{i,j,t}$ with ones on the diagonal.

$$\rho_{i,j,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t}}\sqrt{q_{jj,t}}}$$

Cappiello, Engle and Sheppard (2006) extend the DCC model to allow for asymmetric dynamics in the correlation in addition to the asymmetric response in variances. We can also extend the basic scalar model to allow for asymmetry by adding a single parameter, ξ.

$$Q_t = (1 - \alpha - \beta)\overline{Q} - \xi\overline{N} + \alpha(\varepsilon^*_{t-1}\varepsilon^{*\prime}_{t-1}) + \beta Q_{t-1} + \xi(n_{t-1}n'_{t-1})$$

where $n_t = I\left[\varepsilon^*_t < 0\right] \circ \varepsilon^*_t$ is the asymmetric innovation. $\overline{N} = E\left[n_t n'_t\right]$ with sample analog $T^{-1}\sum_{t=1}^{T} n_t n'_t$.

The updated form can also be constructed on the standardized residuals ε_t as in Jondeau and Rockinger (2006), and both ways give very similar empirical results in our applications. We prefer the second method because it makes the Gaussian rank correlation ρ_{GR} based on η_n directly. Then the conditional covariance of ε^*_{t+1} variables is no longer equal to conditional linear correlation of the raw returns because variance of ε^*_t could be far from one. These "biases" in the sense of linear correlation are not a problem in the framework of the copula which emphasizes the rank correlation. $\varepsilon^*_{t-1}\varepsilon^{*\prime}_{t-1}$ can be interpreted more appropriately as updated information or news shocks, and normalization of the $q_{ij,t}$ variables still insures that the Gaussian rank correlation always falls in the interval from minus one to plus one. Typically these models are usually completed by setting $Q_1 = \overline{Q} = T^{-1}\sum_{t=1}^{T} \varepsilon^*_t \varepsilon^{*\prime}_t$. Another good candidate is the non-target scalar models: $Q_t = h\overline{Q} - \xi\overline{N} + \alpha(\varepsilon^*_{t-1}\varepsilon^{*\prime}_{t-1}) + \beta Q_{t-1} + \xi(n_{t-1}n'_{t-1})$, which converges quickly although the initial $\overline{Q}$ or Q_1 is not well defined.

We could compute the quasi-likelihood function for these models, but in large dimensions, convergence is not guaranteed and sometimes it fails or is sensitive to the starting values. This incidental parameter problem causes quasi-likelihood based inference to have economically important biases in the estimated dynamic parameters, α especially has a serious downward bias. Engle, Shephard and Sheppard (2008) suggest an approach to construct a type of composite likelihood, which is then maximized to deliver the preferred estimator.

$$CL(\psi) = \sum_{t=1}^{T} \frac{1}{M} \sum_{j=1}^{M} \log f(y_{jt}; \psi)$$

where y_{jt} is a pair of data, ψ is a set of parameters, M is the number of all unique pairs, and $t = 1, 2, ..., T$. The composite likelihood is based on summing up the quasi-likelihood of subsets of assets. Each subset yields a valid quasi-likelihood, but this quasi-likelihood is only mildly informative about the parameters. By summing up many subsets we can produce an estimator which has the advantage that we do not have to invert large dimensional covariance matrices. Further, and vitally, it is not affected by the incidental parameter problem. It can also be very fast, and does not have the biases intrinsic in the usual quasi-likelihood when the cross-section is large. We estimate this DCC Gaussian copula by maximizing m-profile subset composite likelihood (MSCL) using contiguous pairs, which is attractive from statistical and computational viewpoints for large dimensional problems compared with the m-profile composite likelihood (MCLE) using all the pairs.

Needless to say, the scalar model is restrictive in forcing the correlation dynamics to be identical across all pairs of assets. We can therefore extend the model to allow for different rank correlation dynamics across several blocks of assets as in Billio, Caporin and Gobbo (2006), Vargas (2006) and Billio and Caporin (2009). These asset blocks can correspond to industries or ratings. We can also estimate it by the Composite Likelihood method, in which different pairs in different blocks should be led by different parameters.

2.1.2 DECO Gaussian Copula

The dynamic equicorrelation (DECO) model in Engle and Kelly (2012) is essentially an extreme case of a DCC model in which the correlations are equal across all pairs of companies but where this common equicorrelation is changing over time. The resulting dynamic correlation can be thought of as an average dynamic correlation between the companies included in the analysis.

We also extend the concept of Dynamic Equicorrelation (DECO) to copula-based models to investigate the residual dependence. Following Engle and Kelly (2012), we parameterize the dynamic rank equicorrelation matrix as

$$R_t = (1 - \rho_t)I_n + \rho_t J_{n\times n}$$

where I_n denotes the n-dimensional identity matrix and $J_{n\times n}$ is an $n \times n$ matrix of ones. The inverse and determinants of the rank equicorrelation matrix, R_t, are given by

$$\begin{aligned} R_t^{-1} &= \frac{1}{(1-\rho_t)}[I_n - \frac{\rho_t}{1+(n-1)\rho_t}J_{n\times n}] \text{ and} \\ \det(R_t) &= (1-\rho_t)^{n-1}[1+(n-1)\rho_t] \end{aligned}$$

Note that R_t^{-1} exists if and only if $\rho_t \neq 1$ and $\rho_t \neq \frac{-1}{n-1}$, and R_t is positive definite if and only if $\rho_t \in \left(\frac{-1}{n-1}, 1\right)$. The time-varying rank equicorrelation parameter, ρ_t is assumed to follow the simple dynamic linear form:

$$\rho_{t+1} = \omega + \alpha u_t + \beta \rho_t + \gamma' y_t$$

where y_t is a vector of exogenous variables, u_t represents the rank equicorrelation update. In our empirical application, we only consider the following updating rule:

$$u_t = \frac{\sum_{i\neq j} \varepsilon_{i,t}^* \varepsilon_{j,t}^*}{(n-1)\sum_i \varepsilon_{i,t}^{*2}} = \frac{\left(\sum_i \varepsilon_{i,t}^*\right)^2 - \sum_i \varepsilon_{i,t}^{*2}}{(n-1)\sum_i \varepsilon_{i,t}^{*2}}$$

Note that u_t lies within the positive definite range $\left(\frac{-1}{n-1}, 1\right)$, and $\varepsilon_t^* = \Phi^{-1}(\eta_n = F_n(\varepsilon_n))$.

The correlation matrices R_t are guaranteed to be positive definite if the parameters satisfy $\omega/(1-\alpha-\beta) \in (-1/(n-1), 1)$, $u_t \in (-1/(n-1), 1)$, and $\alpha + \beta < 1$, $\alpha > 0$, $\beta > 0$. The fitted rank equicorrelation process could obeys the bounds $\left(\frac{-1}{n-1}, 1\right)$ without imposing $\alpha + \beta < 1$. The u_t form above will be the least sensitive to residual volatility dynamics and extreme realizations, owning to the use of a normalization that uses the mean cross-sectional variance.

We can also consider the nonlinear forms similar to those of Dias and Embrechts (2004), such as $\rho_{t+1} = \Lambda(\omega + \alpha u_t + \beta \Lambda^{-1}(\rho_t) + \gamma' y_t)$ or similar to those of Patton (2006 a&b), such as $\rho_{t+1} = \Lambda(\omega + \alpha u_t + \beta \rho_t + \gamma' y_t)$ where $\Lambda(\cdot)$ is the modified logistic function: $\Lambda(x) = \frac{1-\exp(-x)}{1+\exp(-x)}$. The DECO Gaussian copula is estimated by maximizing likelihood functions directly.

2.2 Dynamic t-Copula

The copula of the multivariate standardized t distribution is the t-copula and the dynamic t-copula is defined as follows:

$$C(\eta_1, \eta_2, ..., \eta_n; R_t, v_t) = T_{R_t, v_t}(t_{v_t}^{-1}(\eta_1), t_{v_t}^{-1}(\eta_2), ..., t_{v_t}^{-1}(\eta_n))$$

where $\eta_i = F_i(\varepsilon_i)$ for $i = 1, 2, ..., n$. $\varepsilon_{it} \sim iid(0,1)$, the innovations from the marginal dynamics introduced in the previous section. R_t is the correlation matrix, and v_t is the degrees of freedom. $\varepsilon_i^* = t_{v_t}^{-1}(\eta_i = F_i(\varepsilon_i))$, $t_{v_t}^{-1}(\eta_i)$ denotes the inverse of the t cumulative distribution function. R_t and v_t can be assumed to be constant or a dynamic process through time.

The t-copula generalizes the normal copula by allowing for non-zero dependence in the extreme tails. Since it is a symmetric copula, "upper tail dependence", τ^U, between the variables during extreme appreciations is restricted to be the same as "lower tail dependence", τ^L, during extreme depreciations, and is given by:

$$\tau_t^U = \tau_t^L = 2 - 2T_{v_t+1}(\sqrt{v_t+1}\sqrt{\frac{1-\rho_t}{1+\rho_t}})$$

The normal copula imposes that this probability is zero. The two parameters of the t-copula, ρ_t and v_t, jointly determine the amount of dependence between the variables in the extremes. The coefficient of tail dependence seems to provide a useful measure of the extreme dependence between two random variables. In our multivariate case, we can assume any pairwise tail dependence is determined by the common or average time-varying rank correlations and the degrees of freedom. For the time-varying rank correlations, the tail dependence exhibits time variation even at constant degrees of freedom.

2.2.1 DCC t-Copula

The dynamic processes are assumed to be the same as those defined in the DCC Gaussian copula. We only consider constant degrees of freedom in the DCC t-copula. Since the correlation between Gaussian rank correlation ρ_{GR} and the rank correlation $\rho_{TR} = Corr[t_v^{-1}(u), t_v^{-1}(v))$ for t-copula is almost equal to one, and ρ_{TR} can be well approximated by ρ_{GR}.

We can maximize an objective function that approximates the profile log-likelihood for the degrees of freedom parameter as in Bouye et al. (2000) with the following algorithm:

(1) Let $\widehat{R_t^0}$ be the composite likelihood estimate of R_t for the dynamic Gaussian copula.

(2) R_t^{m+1} is obtained using the following equation

$$\widehat{R_t^{m+1}} = \frac{1}{T}\left(\frac{v_t+N}{v_t}\right)\sum_{t=1}^{T}\frac{[t_{v_t}^{-1}(\eta)]^\intercal[t_{v_t}^{-1}(\eta)]}{1+\frac{1}{v_t}[t_{v_t}^{-1}(\eta)]^\intercal\widehat{R_t^m}[t_{v_t}^{-1}(\eta)]}$$

(3) Repeat the second step until convergence: $\widehat{R_t^{m+1}} \approx \widehat{R_t^m}$.

This method can be significantly faster than maximum likelihood especially for large samples. In our empirical application, we only adopt a two-step algorithm for convenience, which means we first estimate R_t from dynamic Gaussian copula, and then recover the degrees of freedom from t-copula with R_t fixed from the first step.

2.2.2 DECO t-Copula

For the DECO t-copula, dynamic rank equicorrelation parameters are assumed to be the same as those in the DECO Gaussian copula. For the linear form, the update forms for u_t are constructed on the inverse *cdf* of the standard normal distribution: $\Phi^{-1}(\eta_n = F_n(\varepsilon_n))$, and for the nonlinear form on inverse *cdf* of the t distribution: $t_v^{-1}(\eta_n = F_n(\varepsilon_n))$ as in Patton (2006b). Another linear form might be considered, such as:

$$\Lambda^{-1}(\rho) = \omega + \alpha\Lambda^{-1}(u_t) + \beta\Lambda^{-1}(\rho_t) + \gamma^{'}\Lambda^{-1}(y_t)$$

where $\Lambda^{-1}(x)$ is the inverse function of $\Lambda(x) = \frac{1-\exp(-x)}{1+\exp(-x)}$, and the update forms for u_t are constructed on the inverse *cdf* of the t distribution: $t_v^{-1}(\eta_n = F_n(\varepsilon_n))$ directly.

For the dynamic degrees of freedom, Jondeau and Rockinger (2003, 2006) model a dynamic univariate skewed t distribution. We can extend their framework into a dynamic multivariate t-copula:

$$\begin{aligned}\widetilde{v}_{t+1} &= a_1 + \sum_{i=1}^{n} b_{1i}^{+}\varepsilon_{it}^{*+} + \sum_{i=1}^{n} b_{2i}^{-}\varepsilon_{it}^{*-} + c_1\widetilde{v}_t + \gamma^{'}y_t \\ v_t &= g_{[L_v,U_v]}(\widetilde{v}_t)\end{aligned}$$

where v_t is the degrees of freedom, $\varepsilon_{it}^{*+} = \max(\varepsilon_{it}^{*},0)$ and $\varepsilon_{it}^{-} = \max(-\varepsilon_{it}^{*},0)$, y_t is a vector of exogenous variables. This dynamic is mapped into the authorized domain $[L_v, U_v]$ with the logistic function $g(\widetilde{v}_t) = L_v + (U_v - L_v)/(1+\exp(-\widetilde{v}_t))$. There are at least $2n+2$ parameters in this model. To solve this drawback, we can assume that the parameters are the same within the same group, or the same parameters for all.

We consider another class of update forms for the degrees of freedom (DF) based on the log-likelihood function of cross-sectional multivariate t distribution at time t as follows:

$$\begin{aligned}\log(L(v_t)) &= \log(\frac{\Gamma(\frac{v_t+n}{2})}{\Gamma(\frac{v_t}{2})}) - n\log(\frac{\Gamma(\frac{v_t+1}{2})}{\Gamma(\frac{v_t}{2})}) - \\ &\quad \frac{1}{2}\log|R_t| - \frac{v_t+n}{2}\log(1+\frac{\varepsilon_t^{*'}R_t^{-1}\varepsilon_t^{*}}{v_t}) + \frac{v_t+1}{2}\sum_{n=1}^{n}\log(1+\frac{\varepsilon_{nt}^{*2}}{v_t})\end{aligned}$$

Maximizing with respect to the degrees of freedom v_t:

$$DF_t = \arg\max\log(L(v_t))$$

Update forms for u_t constructed on $\Phi^{-1}(\eta_n = F_n(\varepsilon_n))$ are assumed as the instant rank correlation

R_t at time t. The time-varying degree of freedom v_t is assumed to follow:

$$\widetilde{v}_{t+1} = \omega + \alpha DF_t + \beta \widetilde{v}_t + \gamma' y_t$$

$$v_t = g_{[L_v, U_v]}(\widetilde{v}_t)$$

where y_t is a vector of exogenous variables. This dynamic is mapped into the authorized domain $[L_v, U_v]$ with the same logistic function. Since DF_t can be regarded as the instant degrees of freedom at time t, we can keep a linear form directly as:

$$v_{t+1} = \omega + \alpha DF_t^{truncated} + \beta v_t + \gamma' y_t$$

where $DF_t^{truncated}$ is the DF_t truncated at $min(DF_t, U_{DF})$, U_{DF} is the upper boundary to avoid $\alpha = 0$ for sufficiently large data of DF_t. For $v = 50$, the t-distribution is almost the same as the normal distribution, we restrict the authorized domain $[L_v, U_v]$ as $[2, 50]$, and $\omega/(1-\alpha-\beta) > 2$, $\alpha + \beta < 1$, $\alpha > 0$, $\beta > 0$. In our empirical studies, we only consider the linear form $\rho_{t+1} = \omega + \alpha u_t + \beta \rho_t$ with constant degrees of freedom and the linear form of time-varying degrees of freedom.

2.3 Dynamic Grouped t-Copula

In risk management, the tail dependence is very important. At the standard t-copula, the assumption of one global degree of freedom parameter may be over-simplistic and too restrictive for a large portfolio. Empirically we also find that, with more assets, the estimated degrees of freedom could easily become a big number. As in a block correlation dynamic model, we would like to assume different degrees of freedom for different groups corresponding to, for example, industries or ratings.

Consider now the following model. Let $Z_t \sim N_n(0, R_t)$, where R_t is an arbitrary linear correlation matrix, be independent of U, a random variable uniformly distributed on $(0,1)$. Furthermore, let G_v denote the distribution function of $\sqrt{v/\chi_v^2}$. Partition $\{1, ..., n\}$ into m subsets of sizes $s_1, ..., s_m$. Let $R_t^k = G_{v_k}^{-1}(U)$ for $k = 1, ..., m$. If

$$Y = (R_t^1 Z_{1,...,} R_t^1 Z_{s_1}, R_t^2 Z_{s_1+1,...,} R_t^2 Z_{s_1+s_2}, ..., R_t^m Z_n)',$$

then the random vector $(Y_1, ..., Y_{s_1})$ has an s_1-dimensional t-distribution with v_1 degrees of freedom and, for $k = 1, ..., m-1$, $(Y_{s_1+...+s_k+1}, ..., Y_{s_1+...+s_{k+1}})'$ has an s_{k+1}-dimensional t-distribution with v_{k+1} degrees of freedom. The grouped t-copula is described in more details in Daul et al. (2003).

For calibration of and simulation from the grouped t-copula, there is no need for an explicit copula expression. The calibration of this model is identical to that of the t distribution except that the ML-estimation of the m degrees of freedom parameters has to be performed separately on each of the m risk factor subgroups. In our dynamic copula application, we adopt a two-step algorithm for convenience, which means we first estimate R_t from the dynamic Gaussian copula, and then recover v_k degrees of freedom for each group from the t-copula with R_t^k fixed from the first step. The dynamic degrees of freedom can also be built into these grouped t-copulas in the second step, for instance, at group k, the update form for degrees of freedom can be constructed by assuming dynamic rank equicorrelation, then the time-varying degree of freedom $v_{k,t}$ can be estimated by ML-estimation for group k. In our empirical studies, we only consider the constant degrees of freedom for the grouped t-copula.

3 Risk Management Application

Value at Risk (VaR) is probably the most popular risk measure, having a central role in risk management. Although the VaR may be interesting from a practical point of view, however, it has a serious drawback: it does not necessarily satisfy the property of subadditivity, which means that one can find examples where the VaR of a portfolio as a whole is higher than that of the sum of the VaR of its mutually exclusive sub-portfolios. An alternative practically viable risk measurement method that satisfies the subadditivity property and other desirable properties is the expected shortfall (ES).

If a single portfolio is considered, it makes more sense to fit a univariate model to the corresponding return series and base any VaR or ES calculations on this model. On the other hand, if a number of different portfolios based on the same universe of N assets are considered, it is preferred practice to base the individual VaR or ES calculations on a conditional joint multivariate distribution of all N assets. This also allows computing of the marginal risk contributions of each position and evaluation of the effects of hedges. Hence, multivariate distribution is very useful for risk management applications.

When we deal with a large portfolio of assets, however, both VaR and ES estimation can become very difficult owning to the complexity of joint multivariate modeling. Some approaches have been proposed (see Giot and Laurent (2003) for a review), but most of them are rather complicated to implement and can give similar results to simpler methods. Alternatively, simulations play an important role in finance. They are used to replicate the efficient frontiers, to price options, to estimate joint risks, and so on. Using conditional dynamic copulas, it is relatively easy to construct and simulate from multivariate distributions built on marginals and dependence structure. The GARCH-like dynamics in both variance and dependence offer us one-step-ahead predictions of portfolio losses.

We illustrate the following steps for the one-step-ahead simulation:

1. Draw independently $\varepsilon_{t+1}^{*i1}, ..., \varepsilon_{t+1}^{*im}$ for each asset from the n-dimensional normal or t distribution with zero mean, forecast correlation matrix R_{t+1}, or degrees of freedom v_{t+1}, and obtain

$\mu_{t+1}^{i1}, ..., \mu_{t+1}^{im}$ by setting $\mu_{t+1}^{ik} = \Phi(\varepsilon_{t+1}^{*ik})$ or $t_{v_{t+1}}(\varepsilon_{t+1}^{*ik})$,where $k = 1, ..., m$, the total paths of simulation, $i = 1, ..., n$, the number of assets;

2. Obtain $\varepsilon_{t+1}^{i1}, ..., \varepsilon_{t+1}^{im}$ by setting $\varepsilon_{t+1}^{ik} = F_i^{-1}(u_{t+1}^{ik})$,where F_i is empirical marginal dynamics distribution for asset i;

3. Obtain $z_{t+1}^{i1}, ..., z_{t+1}^{im}$ by setting $\varepsilon_{t+1}^{ik} * \sigma_{t+1}^{i}$,where σ_{t+1}^{i} is the forecast standard deviation by GARCH model for asset i;

4. Obtain $X_{t+1}^{i1}, ..., X_{t+1}^{im}$ by setting $X_{t+1}^{ik} = \lambda_{t+1}^{i} + z_{t+1}^{ik}$,where λ_{t+1}^{i} is the forecast mean by $ARMA$ model for asset i;

5. Finally obtain the portfolio return $r_{t+1}^{1}, ..., r_{t+1}^{m}$ by setting $r^k = [X_{t+1}^{k}]^{\intercal} [W_{t+1}]$, where W_{t+1} is the portfolio weights at $t + 1$.

The VaR and ES can be obtained by these simulated $r_{t+1}^{1}, ..., r_{t+1}^{m}$ portfolio returns. The multi-day ahead VaR and ES can also be obtained by simulating multi-day forward in a similar way.

In our empirical study, we compare the effects of these copulas on risk measures VaR and ES by backtesting. For the unconditional coverage testing, VaR performance is acessed by Kupiec's (1995) LR test, LR_{uc} and a simple mean test, $MT = \sqrt{T}\frac{\pi - p^*}{\sqrt{Var(I_t)}} \sim N(0, 1)$ where I_t is "hit sequence", equal to one when loss returns in excess of VaR and zero otherwise. As for the independence testing, it is done by Ljung-Box test, LB, and Christoffersen's (1998) LR test, LR_{CC}. We evaluate ES performance by ES bootstrap test of McNeil and Frey (2000).

3.1 Data

The conditional dynamic copulas presented are applied to a portfolio composed by all companies listed on CDX.NA.IG (a credit default swap index which contains 125 North American investment grade companies). The database contains 2500 daily returns over the period from Jan 03, 1995 until Dec 03, 2004 taken from the CRSP database. This set includes 125 companies although 36 have one or more periods of non-trading, for example prior to IPO or subsequent to an acquisition. Selecting only the companies that have returns throughout the sample reduces this set of 89. The firms selected are reported in Table 11. The data set covers the high-tech bubble period, including the post high-tech bubble period, but just before the housing bubble. From the return dispersion, we find dispersion shrinks a little around the end of 1998, and there is a clear dispersion pattern change within 2001 to 2005, the post high-tech bubble period. Thus, this sample is perfectly suitable for highlighting the importance of the copula-based dependence approach compared with the traditional correlation analysis.

3.2 Empirical Results

In this section we first use the full sample to show all model differences. The marginal dynamics is selected, and the dynamic dependence is tested. Results of the estimation of the copula models described in the previous section are presented in several figures and tables. We then compare the out-of-sample performances in the post high-tech bubble period not only for passive large portfolios but also for dynamic optimal large portfolios by both Mean-Variance and ES criteria.

3.3 In-Sample Analysis

For the marginal dynamics, we simply fit a first-order autoregressive model to the conditional mean of the returns of each equity $r_{i,t} = C_i + \Phi_i r_{i,t-1} + z_{i,t-1}$, and a symmetric GARCH model to the conditional variance $\sigma_{i,t} = K_i + \alpha_i z_{i,t-1}^2 + \beta\sigma_{i,t-1}$, the standardized returns $\varepsilon_i = z_{i,t}/\sigma_{i,t}$. We estimate this $AR(1){-}GARCH(1,1)$ by the Quasi-Maximum Likelihood. We also try $ARMA(p,q){-}GARCGH(1,1)$ for each return, and let AICC select the best model. The results are as robust as those obtained by fitting $AR(1) - GARCH(1,1)$ to all returns. After the conditional mean and volatility clustering have been filtered out, as shown by Figure 1, these standardized residuals look like $iid(0,1)$ but with a non-normal distribution. Largely the kurtosises are quite high and skewnesses are still different from zero. Apparently the Pearson's linear correlation is not a good measure of dependence even for these standardized residuals.

We further test the dynamic dependence with the BQ test (Busetti and Harvey (2011)). These tests do not require a model for the copula but can be regarded as stationarity tests for time-varying bivariate quantiles. Consider a bivariate series, y_{1t} and y_{2t}, $t = 1, ..., T$, by converting to ranks we can obtain the sample quantiles and the empirical copula. The empirical copula yields the proportion of cases in which both observations in a pair are less than, or equal to, particular $\tau - th$ quantiles: $\overline{\xi}_1(\tau)$ and $\overline{\xi}_2(\tau)$. This proportion will be denoted as $C(\tau,\tau)$. The value of $C(\tau,\tau)$ indicates the strength of dependence at τ. There are six test statistics here: four single bi-quantic tests (BB, AA, BA, and AB), a combined bi-quantic test, and a quadrant association test. BB is constructed from the proportion of cases $(y_{1t} \leq \overline{\xi}_1(\tau), y_{2t} \leq \overline{\xi}_2(\tau))$, AA from $(y_{1t} \geq \overline{\xi}_1(\tau), y_{2t} \geq \overline{\xi}_2(\tau))$, BA from $(y_{1t} \leq \overline{\xi}_1(\tau), y_{2t} \geq \overline{\xi}_2(\tau))$, and AB from $(y_{1t} \geq \overline{\xi}_1(\tau), y_{2t} \leq \overline{\xi}_2(\tau))$. A combined test is based on any three proportions from above, whereas the quadrant association test is based on the sum of the proportion of cases $(y_{1t} \leq \overline{\xi}_1(\tau), y_{2t} \leq \overline{\xi}_2(\tau))$ and $(y_{1t} \geq \overline{\xi}_1(\tau), y_{2t} \geq \overline{\xi}_2(\tau))$. Under the null hypothesis of serially independent observations with a constant copula $C(\tau,\tau)$ over time, the asymptotic distribution of these stationarity test statistics is Cramer von Mises. The 1%, 5% and 10% critical values are 0.743, 0.461 and 0. 347 respectively, except for the combined test (at three-degree freedom) where they are 1.359, 1.000 for 1%, 5%. The 10% critical value for the combined test is not available. Table 1 reports rejection ratios by BQ test statistics on those 3916 pairwise standardized returns. The rejection ratio is the ratio of the number of pairwise over critical value to the total number of pairwise. Tests associated with different quantiles point to changes in different parts of the copula. We can see that these copulas are not constant through time on average, and even at these lower critical values, the rejection ratios are still high.

In Figure 2, we plot the dynamic dependences for DCC Gaussian copulas from January 03, 1995 through to December 03, 2004. The rank correlation dispersion from the DCC Gaussian copula looks very dynamic in the range of [-0.3 0.8]. Figure 3 plots the average correlation from DECO/DCC, and the average rank correlation ρ_{GR} from DCC/DECO Gaussian copulas. The equicorrelations from DECO actually are very similar to the average pairwise correlations from DCC, whereas the average pairwise rank correlations from the DCC copula are a little lower than the rank equicorrelations from the DECO copula, which might be caused by assumption of $Q_1 = \overline{Q} = T^{-1}\sum_{t=1}^{T} \varepsilon_t^* \varepsilon_t^{*\prime}$. The non-target scalar models could do a better job here. The correlations from DECO/DCC are a little lower, about 0.04 on average, than rank correlations ρ_{GR} from their counterpart copula models. As we know, linear correlation depends on both marginal distributions and copulas, if true distributions have fat tail dependence; for example, in the t-copula, the derived rank correlation could be higher than linear correlation. Therefore this additional 0.04 could be from the marginal distributions and tail dependence which the DECO/DCC model neglects. As shown by Figure 4, the equidependence from the DECO t-copula with constant degrees of freedom (DF) looks a little more smooth than that from the DECO t-copula with dynamic DF, but still similar. The likelihood ratio test rejects the null hypothesis of constant DF. The DF is negative correlated with rank correlations at -62.5%, which means the dynamic process of DF could recover more information about fat tails; for example, at the end of 2002, the rank correlation is high, whereas the DF is still heading down. The tail dependence is high around 2002-3, which reveals that the high-tech bubble crush has more systemic risk effects.

The estimators are reported in Table 2. For the t-copula with a single DF parameter, the DF are all similar around 36, and the dynamic DF fits the t-copula better than the constant DF by the log-likelihood. For grouped t-copula by industries, we find a lot of discrepancies of DF among these eight industry sections. The average DF at DCC grouped t-copula is down to 25.534 with lowest DF at 13.906 in utility section, whereas for DECO grouped t-copula, these numbers become 15.234 and 4.511. Therefore the assumption of one global DF parameter could be over-simplistic and too restrictive for a large portfolio. For our DECO grouped t-copula, we assume equidependence as one group instead for each industry section, so the resulting DF at each section will be affected by this assumption. We expect a DECO grouped t-copula in which both DF and equidependence are grouped would work better.

3.4 Passive Portfolios and Risks (Out-Sample Tests)

In our tests, we consider both an equally-weighted portfolio and a value-weighted portfolio. For the value-weighted portfolio, the weights are updated annually by their market capitalizations. The upper plot of Figure 5 shows the average value-weights over our sample testing period from Dec 08, 2000 through to Dec 04, 2004. As we see, the value-weights are not well-balanced. General Electric Co. alone weights up to 13.4%, which would affect the portfolio forecasting considerably. The equally-weighted portfolios and value-weighted portfolios typically represent two kinds of portfolios: well-balanced and not well-balanced portfolios, suitable for testing the proposed copulas. Furthermore the value-weighted portfolio is not static but still updated annually.

In this example, we initially estimate the parameters of models using data from t_1 to t_{1500}, the first 1500 data from Jan 03, 1995 to Dec 07, 2000, and backtesting starts from Dec 08, 2000 through to Dec 04, 2004 which covers the post high-tech bubble period for a total of 1000 out-of-sample observations. We simply simulate 2000 values of the innovations for each asset, which, as we have observed, provides a rough approximation to the observed distribution of model residuals, then evaluate the VaR and ES value at lower quantiles as $\alpha = 0.005$, 0.01, 0.05 and 0.1 at equal weights and value weights by given w_{t+1}. For the next 49 observations we use this same estimated model, but the marginal distributions are updated daily. That means we use the same copula simulation for each 50 observations, but at each new observation we update the VaR and ES estimates. We re-estimate the model and repeat the whole process until all days have been included in the estimation sample from t_{1501} to t_{2500} for a total of 1000 out-of-sample observations. The realized portfolio returns at the w_{t+1} are recorded for evaluation.

In order to better show the performance of the dynamic copula, we start from two typical non-copula methods and constant copula subsequently. Meanwhile, we also examine if equal correlation or dependence performs about as well as full dependence structure in each model.

3.4.1 Non-Copula

First, we compute the one-day-ahead VaR and ES from DCC/DECO by assuming the standardized residuals from multi-normal distribution and multi-t distribution. The resulting VaR and ES are based on simulations directly from multi-normal or multi-t distribution. Table 3 reports p-values for unconditional coverage tests, independence tests, and ES bootstrap tests on these two weighted portfolios. Compared with DECO, DCC does a rather better job especially for the value-weighted portfolio by recovering more dispersed correlation structure. For not well-balanced portfolios, assuming equal correlation apparently is too restrictive, especially in our case, General Electric Co. alone weighs up to 13.4%. The results improve a little from multi-normal distribution to multi-t distribution with degrees of freedom at 72 for DECO and 62 for DCC. The hitting ratios, however, are still too high. Clearly, simulation directly from multi-normal distribution or multi-t distribution with such high degrees of freedom would not induce enough fat tail.

In order to better fit the tails of the return distributions and to match the theoretical VaR levels, we can also obtain the one-day-ahead VaR and ES by assuming the portfolio returns are from univariate conditional t-distribution, but use the estimated conditional covariance matrix from DCC and DECO to compute variance of portfolio returns. To be more specific, the estimated conditional variance of the portfolio at time t is given by $\widehat{h}_{w,t} = w'\widehat{H_t}w$, where w is the vector of weights, and $\widehat{H_t}$ is estimated covariance from DCC or DECO. At time $t-1$, conditioning on the past portfolio returns and their corresponding estimated conditional variances, we can choose the number of degrees of freedom, v^*, that maximizes the likelihood

$$\prod_{s=1}^{t-1} \frac{\Gamma(\frac{v+1}{2})}{\sqrt{\pi(v-2)\widehat{h}_{w,s}\Gamma(v/2)}}(1+\frac{(w'x_s)^2}{(v-2)\widehat{h}_{w,s}})^{-(v+1)/2}$$

over v, where $\Gamma(\cdot)$ denotes the gamma function. Note that the standard formula for the t-distribution has been modified by the scale factor $\widehat{h}_{w,s}(v-2)/v$, where the degrees of freedom adjustment is designed so that $\widehat{h}_{w,s}$ is exactly equal to the conditional variance of $w'x_s$. v^* having been thus found, the α quantile VaR at time t is finally computed as $t_{v^*,\alpha}\sqrt{\widehat{h}_{w,t}(v^*-2)/v^*}$, where $t_{v^*,\alpha}$ denotes the α quantile of the t-distribution with v^* degrees of freedom. To compare with other models, we compute VaR and ES from simulation instead of by theoretical formula.

As before, shown by the bottom section of Table 3, compared with DECO, DCC does a rather better job especially for the value-weighted portfolio by taking care of more dispersed correlation structure. ES bootstrap tests and independence tests are all doing very well. The univariate t-distribution indeed recovers more fat tail, but sacrifices the upper quantiles, and, as we see, the hitting ratios at $\alpha = 0.05, 0.1$ are still too high. By DECO or DCC, in multivariate or univariate distribution, the main information lost is the marginal distribution. So the marginal distribution is very important for VaR and ES, and making the marginal distribution endogenous requires a copula.

3.4.2 Constant Copula

Next, we consider Constant-Conditional-Correlation (CCC). Bollerslev (1990) suggested a multivariate GARCH (1,1) model where the conditional correlations are constant over time. In the framework of the copula, we assume $R_t = R$, the full constant dependence matrix, and $\rho_t = \rho$ which is the average of all pairwise rank correlations of R. A problem with this model is the assumption of a constant conditional rank correlation, which conceivably will not always hold. In our sample testing period from Dec 08, 2000 through to Dec 04, 2004, there is a sharp $\frown$ −shaped dependence pattern change from around 12% in early 2001 up to 40% at the end of 2002, and down back to 25% in late of 2004. Table 4 gives us the ex ante VaR and ES with equal dependence and full dependence matrix from the constant Gaussian copula and the t-copula respectively. Overall, the hitting ratios are too high. It seems that neglecting dynamic dependence alone would cause serious problems in risk management.

3.4.3 Dynamic Copula

Last, we explore in detail conditional dynamic copulas. Our VaR and ES are only based on 2000 simulated values of the innovations. In order to fix the sample bias, we also apply extreme value theory to those simulated portfolio returns. We estimate the empirical *cdf* of these forecast portfolio returns with a Gaussian kernel, and a parametric generalized pareto distribution to those returns that fall in each tail at upper and lower thresholds of 10%. Then We compute VaR and ES on the empirical *cdf* of these forecasted portfolio returns. The method can recover more fat tail information, especially at lower quantiles $\alpha = 0.005, 0.01$.

Table 5 reports p-values for unconditional coverage tests, independence tests and ES bootstrap tests from DECO/DCC Gaussian copula respectively. These models are working very well in both VaR and ES. The DCC Gaussian copula does a rather better job especially for the value-weighted portfolio by taking care of more dispersed rank correlation structure. The semi-parametric form of these forecast portfolio returns distributions further adjusts VaR and ES as well.

Table 6 reports the testing results from DECO and DCC t-copulas respectively. The results are very similar to those from the Gaussian copula, partly because in our two-step estimation, the correlation matrix R_t is actually given by the Gaussian copula at the first step, but the degrees of freedom recovered at the second step are on average still quite high at 40. As shown by Table 7, the grouped t-copulas give us a marginal improvement at the lower quantiles $\alpha = 0.005, 0.01$, for example from DCC t-copula at 0.17 (equally-weighted) to DCC grouped t-copula at 0.16 when $\alpha = 0.01$, and from 0.01 to 0.007 when $\alpha = 0.005$. We only have 1000 out-of-sample observations, which might not be enough to reveal fully the benefit of this grouped t-copula.

We could also estimate the degrees of freedom by a rolling window. For example, take a half year or one year recent window. As before, we could fix the correlation matrix R_t estimated from the Gaussian copula at the first step. Usually in this way, at least in our sample, the degrees of freedom are much lower than those estimated by a full window. There are some problems, however, with this rolling method: the selection of width of window is very subjective; the degrees of freedom usually rely on extreme events which do not happen often, and excluding these outliers by a rolling window might not be helpful to recover the fat tails, and the loss could be underestimated accordingly. It seems that fully modeling the dynamic of degrees of freedom is a better method than simple rolling. The only model we consider empirically in this paper is a DECO t-copula with dynamic DF. As shown in the full sample, the DF is negative correlated with rank correlation at -62.5%, which means the dynamic process of DF, by offering another dimension of freedom, could recover more information about fat tails. The bottom section in Table 6 reports the test results from the DECO t-copula with dynamic DF. Compared with the DECO t-copula at a constant DF, there is a marginal improvement at the lower quantiles $\alpha = 0.005, 0.01$. We can also build the dynamic DF into DCC/DECO grouped t-copulas, which are not explored empirically here: intuitively these DCC/DECO t-copulas with grouped dynamic DF will match the fat tails further.

Table 8 reports the out-of-sample log likelihoods and the p-values of the pair-wise Q test under the null hypothesis of equal predictive accuracy of two alternative copula specifications. We define the empirical copula vector $\widehat{U}_{t+1}$ associated with the forecast errors $\widehat{\varepsilon_{t+1|t}}$. Let $d_{t+1} = \log \widehat{c}_{A,t}(\widehat{U}_{t+1}) - \log \widehat{c}_{B,t}(\widehat{U}_{t+1})$ denote the difference in log scores for the two competing copulas A and B, and assume that N out-of-sample observations are available. According to Diks, anchenko and Dijk (2010), the test for equal log scores is obtained as

$$Q = \sqrt{N}\frac{\overline{d}_N}{\widehat{\sigma}_N}$$

where $\overline{d}_N$ is the sample mean of the log score difference d_{t+1}, and $\widehat{\sigma}_N$ is an autocovariance and heteroskedasticity consistent (HAC) estimate of the asymptotic standard deviation of $\sqrt{N}\overline{d}_N$. Under null hypothesis that equal predictive accuracy of two alternative copula specifications, Q is asymptotically standard normally distributed. We find that the t-copula is favored over Gaussian, and grouped t-copula is favored over t-copula for both DECO and DCC copulas.

3.5 Portfolio Optimization and Risks (Out-Sample Tests)

An equally important application of the conditional dynamic elliptical copula is portfolio optimization. When asset returns are far from being normally distributed, the traditional mean-variance framework may produce misleading results. In this section we first minimize the portfolio's ES by the scenario-based copula models and compare its performance against tail risk with that by the traditional mean-variance framework, and then explore the risk management for optimal portfolios accordingly.

3.5.1 Portfolio Optimization

The conditional covariance matrix from elliptical copula can be an input to the Markowitz (1952) portfolio selection method and, as we discussed in previous sections, the rank correlation from an elliptical copula could be more robust than linear correlation. Hence, we examine the gains from diversification obtained by taking into account the time-changing nature of the rank covariance matrix. The classical optimization problem of a portfolio manager is the following one:

$$\sigma^2_{p+t+1} = \min_{w_{t+1}} w'_{t+1}\Sigma_{t+1}w_{t+1}$$

$$
\begin{aligned}
s.t.: \quad & w'_{t+1}1 = 1 \\
& w'_{t+1}\mu_{t+1} = g
\end{aligned}
$$

where g is the portfolio target return; $w_{t+1}$is the optimal vector of weights at time $t+1$; Σ_{t+1} is the forecast variance-covariance matrix of asset returns at time $t+1$; μ_{t+1} is the vector of expected returns at time $t+1$.

In order to avoid having to specify the vector of conditional expected returns, which is more a task for the portfolio manager than a statistical problem, we focus on constructing the global minimum variance portfolio by minimizing $w'_{t+1}\Sigma_{t+1}w_{t+1}$ s.t. $w'_{t+1}1 = 1$. It is well known that optimizers actually act as statistical "error maximizers". There are a lot of approaches like shrinkage estimation, portfolio constraints, Bayesian decision, and robust statistics which effectively reduce or eliminate the influences of outliers in the data, and get more balanced estimated expected return and covariance. Here, we simply do not allow shorting. Jagannathan and Ma (2003) show that constraining portfolio weights to be non-negative is equivalent to shrinkage by reducing the extreme portfolio weights that are associated with estimation error. Empirically we also find this non-negative constrained portfolio works better than unconstrained portfolio by global minimum variance.

When asset returns are far from normally distributed, the traditional mean-variance framework may produce misleading results. In fact, it has been demonstrated that portfolios efficient with respect to the traditional measure of risk, such as the variance, are not efficient with respect to the tail risk measures. To monitor and control the tail risk measure, we also minimize the portfolio's ES by the scenario-based model proposed by Rockafellar and Uryasev (2000), and Andersson et al. (2000):

$$
F_{\beta(w_{t+1},k)} = k + \frac{1}{(1-\beta)N}\sum_{i=1}^{N}(-r_{i,t+1}w_{t+1} - k)^{+}
$$

where $r_{i,t+1}$ is the return in percentage for scenario $i = 1, ..., N$, k is the threshold value at quantile of β. We can reduce this optimization problem to the linear programming problem as:

$$
\begin{aligned}
& \min_{w,z,k} k + \frac{1}{(1-\beta)N}\sum_{i=1}^{N} z_i \\
s.t. \quad : \quad & z_i \geqslant -r_{i,t+1}w_{i+1} - k,\, z_i \geqslant 0, w'_{t+1}1 = 1,\, w'_{t+1}\mu_{t+1} = g
\end{aligned}
$$

where $z_i = (-r_{i,t+1}w_{i+1} - k)^{+}$. In order to compare with the global minimum variance portfolio directly, we assume the conditional expected returns are zeros instead of $AR(1)$.

The simulation algorithm is the same as that in the previous section except the next optimal weight w_{t+1} is estimated by minimizing variance-covariance matrix Σ_{t+1} directly, or ES based on those 2000 simulated values. Table 9 reports the statistics for the four realized weighted portfolio:

optimal portfolio by Mean-Variance and ES criteria from December 08, 2000 through to December 03, 2004. As we expect, the optimal portfolio by DCC copula gives us lower portfolio variance, VaR and ES at 10% quantile than the DECO copula. Although Elton, Gruber, and Green (2007) find that, despite all the advances in modeling, the assumption that all correlations are identical to each other still performs about as well as any other forecasting technique for portfolio optimization, here we find that the modeling dependence dispersion correctly can still make a marginal improvement in portfolio optimization, which should not be neglected casually. For ES at quantile 10%, the optimal portfolio by ES does a better job in both DECO and DCC copulas, whereas for standard deviation, the optimal portfolio by Mean-Variance does a slightly better job only in the DCC copula. So the Markowitz (1952) portfolio selection is not efficient if we consider the tail risk. We also tried $AR(1)$ expected returns, however, the ES at quantile 10% from the scenario-based model could not beat those from global minimum variance portfolio. The discrepancies could come from the expected returns. Chopra and Ziemba (1993) argued that correctly estimating expected returns is ten times more important than getting the variances right, and correlations are even less important. So the estimation errors on expected returns are critical to the optimal portfolio by ES.

In Figure 5, we can find the average optimal weight differences among DECO/DCC Gaussian copulas and t-copulas. ES is also affected by the overall risks. The weight patterns (selected vs. non-selected) are similar, but the magnitudes for some firms are greatly affected by the models, and at any time t the weights can also be very different. To evaluate the performances of all these models rigorously, we have to control the expected returns, and try more non-normal financial asset data.

3.5.2 Optimal Portfolio Risk Management

Because optimizers actually act as statistical "error maximizers", even not allowing short-selling, the ex ante VaR and ES for optimal portfolios are still too low for DCC copula, however, for the DECO copula, the results are not too bad as shown at Table 10. It seems that assuming equicorrelation can help to get more balanced estimated covariance matrix. Another way to deal with this optimal portfolio is to treat DF as an exogenously given variable, which is also mentioned in Nystrom and Skoglund (2002a & b). We might adjust this VaR without simulation as

$$VaR_{p,t+1} = w_{t+1}^{'}\mu_{t+1} + t_{v,\alpha}^{*}(w_{t+1}^{'}\Sigma_{t+1}w_{t+1})^{1/2}$$

where Σ_{t+1} is the covariance matrix derived from the DECO/DCC Gaussian copula, $t_{v,\alpha}^{*}$ is the α% quantile of a standardized t distribution with ν degrees of freedom, and $t_{v,\alpha}^{*}$ is exogenously given. It is clear that this approach does not emphasize the marginal distributions, and $t_{v,\alpha}^{*}$ can be adjusted flexibly enough to cover the misspecification of conditional expected return. We can also construct the global minimum variance portfolio by DCC/DECO without specifying the marginal distributions. Since the comovement of correlations from DCC/DECO is very similar to those from DCC/DECO Gaussian copulas, the resulting optimal portfolio weights are also very similar, only with different $t^*_{v,\alpha}$ to adjust the levels. Correlation constructed by DCC/DECO, however, is Pearson's linear correlation, as we discussed in the previous sections, and linear correlation depends on both the marginal distributions and the copula. It is not a robust measure, and a single observation can have an arbitrarily high influence on it. It seems that rank correlation from a Gaussian copula would be more stable. Tables 10 report p-values for unconditional coverage tests, independence tests and ES bootstrap tests on our optimal weighted portfolio. We chose $DF = 15$ for the DECO copula and $DF = 7$ for the DCC copula. This method is simple and actually works well, especially at lower quantiles 1% and 0.5%. VaR violations still cluster, however, at the upper quantiles of 10% and 5%, partly because we neglect the marginal distributions in this approach. We fix $t^*_{v,\alpha}$ through time, which means we adjust the level instead of comovement. It seems that there is not much "comovement" misspecification compared with "level" misspecification in our dynamic copula models, otherwise, the hitting ratios at certain quantiles would go wildly adrift whatever exogenous degrees of freedom we tried. Modeling the expected returns well could make our simulation in the copula framework work better.

4 Conclusion

The aim of this paper has been to introduce a family of more general Dynamic Conditional Elliptical Copulas for large-dimensional optimal portfolio allocation and risk management. These suggested copulas combine some recently developed techniques in the areas of Multivariate GARCH (DCC/DECO), the Composite Likelihood method, and Extreme Value Theory.

We propose a series of dynamic copulas to examine VaR and ES empirically not only for passive large portfolios but also for dynamic optimal large portfolios by both Mean-Variance and ES criteria. On the basis of empirical studies on portfolios made up of 89 stocks from CDX-listed firms between 1995 and 2005, we find: (1) Modeling the marginal distribution well is very important. DCC or DECO does not specify the marginal, and the fat tail recovered by DCC/DECO is limited. (2) Neglecting dynamic dependence alone in the copula can cause over-aggressive risk management, and the proportion of excessive losses can be seriously underestimated especially when there is a big dependence pattern change. (3) The DECO/DCC Gaussian copula and t-copula work very well in both VaR and ES. The DCC copula is necessary for the highly unbalanced portfolio. For well-balanced portfolios, like the equally-weighted portfolio, the DECO copula still performs about as well as the DCC copula. (4) Grouped t-copula and full dynamic t-copula in which the degrees of freedom are also modeled dynamically can match the fat tail further with an improvement especially at the lower quantiles. (5) Modeling dependence dispersion structure correctly can still make a marginal improvement in portfolio optimization, which should not be neglected. The optimal portfolio by ES does a better job than the optimal portfolio by mean-variance in both DECO and DCC copulas; the Markowitz (1952) portfolio selection is not efficient if we consider the tail risk. (6) Portfolio optimization can induce statistical error maximization. Assuming multivariate t innovations with an exogenously given degree of freedom is a flexible and applicable method for optimal portfolio risk management.

The results also suggest a number of extensions. We present a series of copula models which are not fully explored in our empirical studies, such as other dynamic update processes for both copula dependence and degrees of freedom, and their block counterparts. Except for the semi-parametric form of the marginal distribution which allows asymmetry distribution, both the $ARMA{-}GARCH(1,1)$ and elliptical copulas considered empirically are very simple symmetric parametric models. Furthermore, minimizing the portfolio's ES depends on simulation; if allowing short-selling, symmetric copula would easily induce over-conservative VaR and ES estimates. As we know, the upper tail dependence is usually different from the low tail dependence (see, for example, Ang, Chen and Xing (2006)), and asymmetric models could be critical in optimal portfolio allocation and active risk management. We would like to consider the conditional asymmetric dynamic process in variance and dependence, plus a skew t-copula to explore further various types of financial risks. Since the DECO elliptical copula has a single parameter describing overall dependence, other copulas in the Archimedean class, e.g. the Clayton n-copula, and a mixed-copula approach will be considered for later comparative studies.

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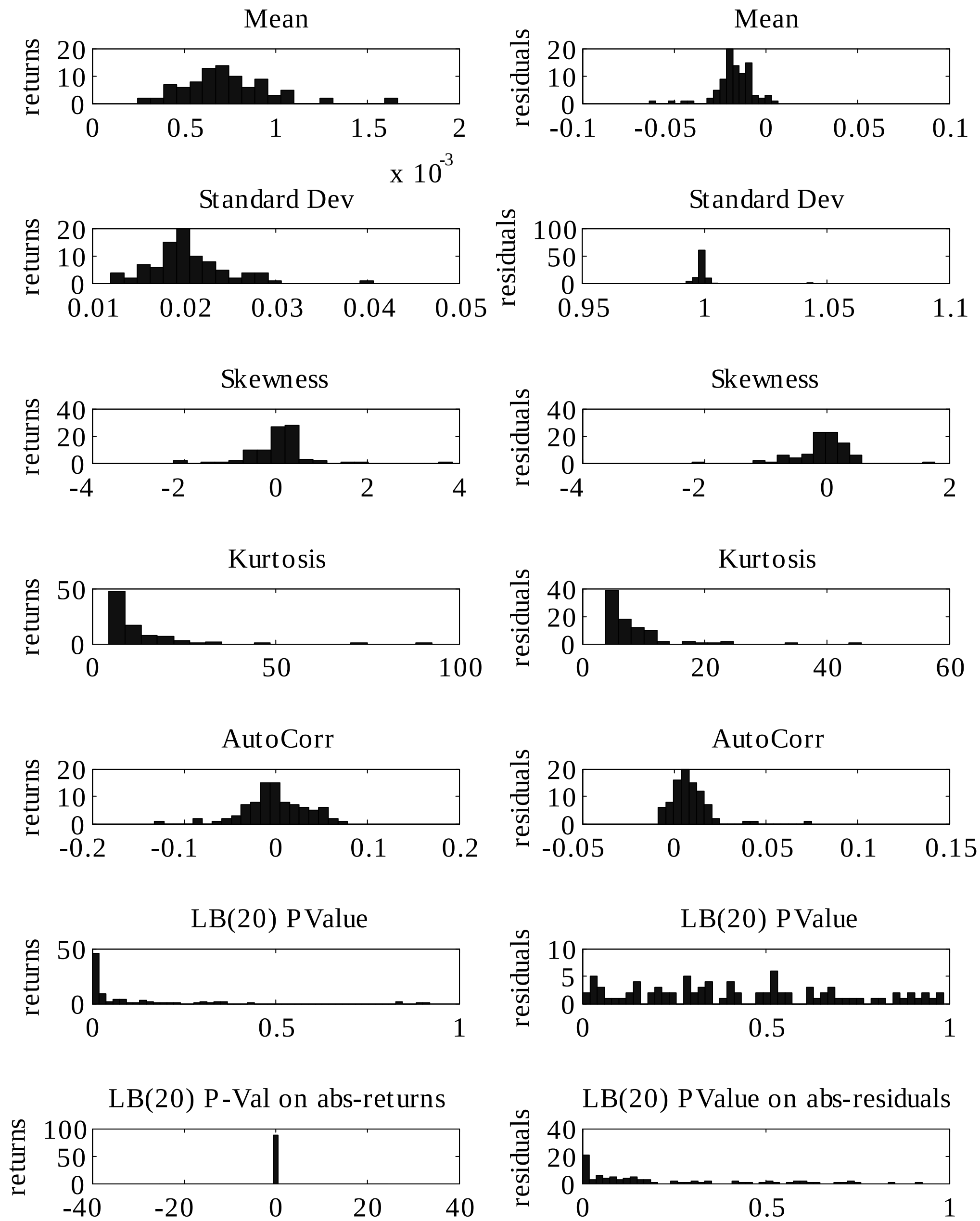
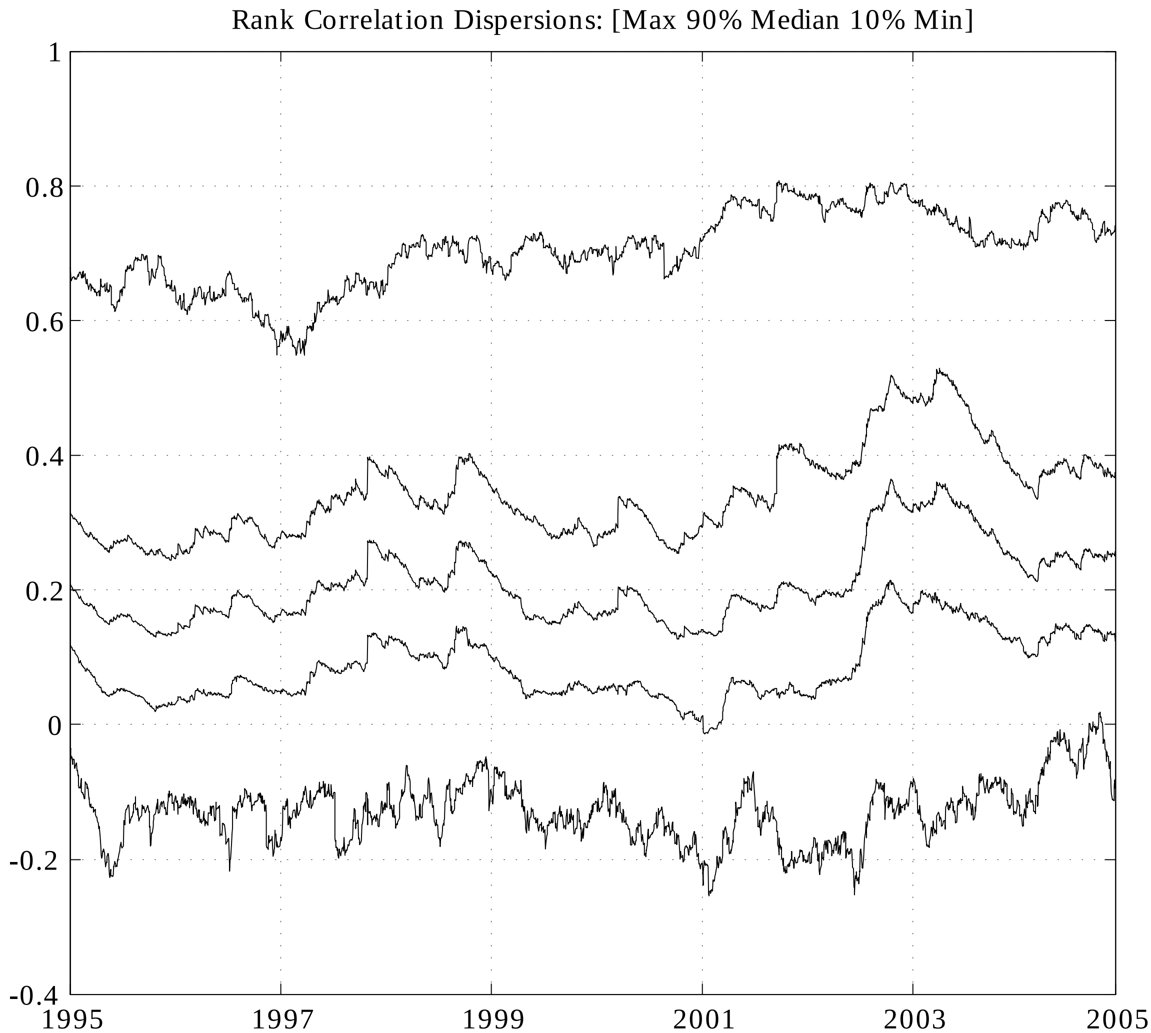
Figure 1: Descriptive Statistics for Returns and Standardized ARIMA-GARCH Residuals



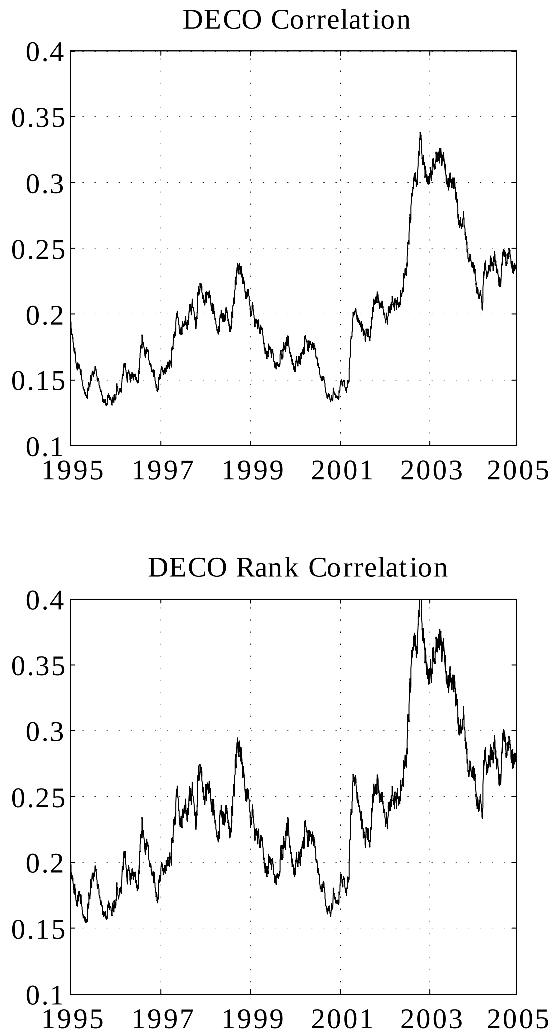
Notes to Figure: We report histograms of the first four sample moments and first-order autocorrelations of the returns and standardized ARIMA-GARCH residuals of the 89 CDX companies. We also report the p-values from a Ljung-Box test where the first 20 autocorrelations are zero for the residuals and absolute residuals. The sample period is from Jan 03, 1995 to Dec 03, 2004.

Figure 2: Rank Correlation Dispersions from DCC Gaussian Copula



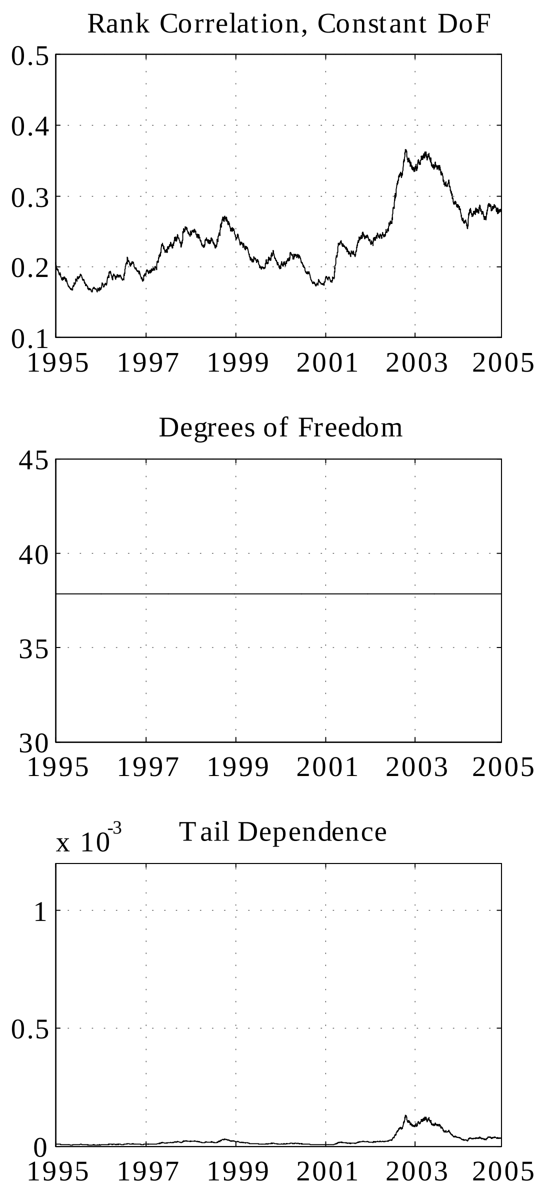
Notes to Figure: We report the rank correlation dispersions from a DCC Gaussian copula (the symmetric scalar model) for the 89 CDX companies from January 03, 1995 to December 03, 2004. At each time t, the correlation dispersion is constructed by Max, quantiles (90%, Median, 10%) and Min of 3916 pairwise equity returns. The DCC Gaussian copula is estimated by maximizing m-profile subset composite likelihood (MSCL, contiguous pairs).

Figure 3: Average Dynamic Conditional Dependences



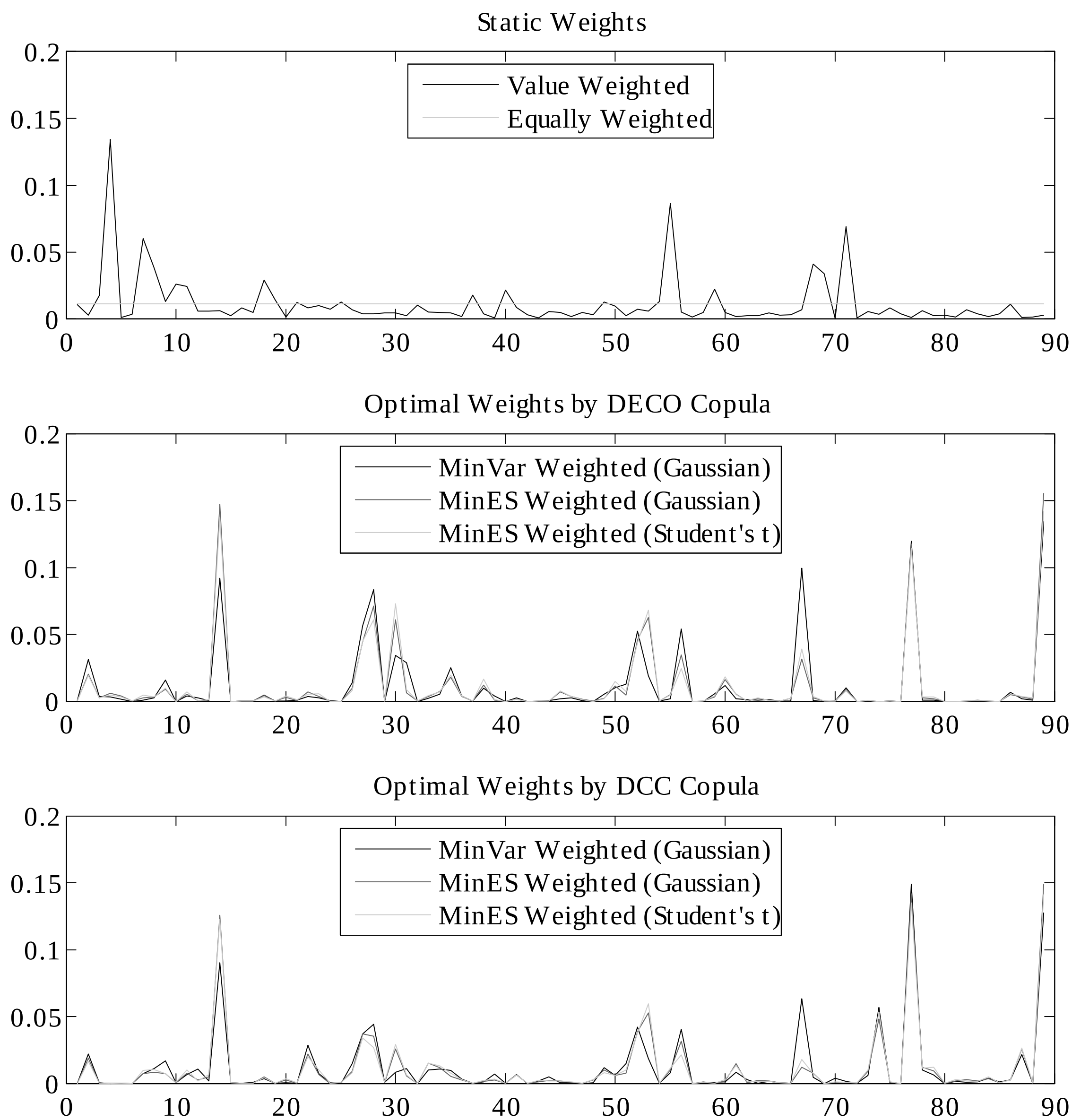
Notes to Figure: We report average correlations from DCC/DECO, and average rank correlations from DCC/DECO Gaussian copulas for the 89 CDX companies from January 03, 1995 to December 03, 2004.

Figure 4: Dependences from DECO t-Copulas, January 03, 1995 to December 03, 2004



Notes to Figure: We report the rank equicorrelations, degrees of freedom, and tail dependences from DECO t-copulas for the 89 CDX companies from January 03, 1995 to December 03, 2004.

Figure 5: Average Portfolio Weights



Notes to Figure: We report the average static and optimal weights for the 89 CDX companies from December 08, 2000 to December 04, 2004. The value-weights are updated by the market capitalizations annually, and the optimal weights are derived by DECO/DCC copulas.

Table 1: Rejection Ratios by BQ Test Statistics, January 03, 1995 to December 03, 2004

Quantiles	0.05	0.10	0.25	0.50	0.75	0.90	0.95
Single bi-quantic test			At 10% Critical Value				
BB	0.12	0.32	0.55	0.34	0.38	0.47	0.45
AA	0.41	0.53	0.49	0.53	0.42	0.27	0.14
BA	0.25	0.29	0.34	0.48	0.44	0.38	0.33
AB	0.26	0.34	0.38	0.50	0.44	0.38	0.31
Combined bi-quantic test*							
Quadrant association test	0.40	0.48	0.53	0.70	0.57	0.52	0.47
Single bi-quantic test			At 5% Critical Value				
BB	0.06	0.21	0.43	0.24	0.27	0.34	0.32
AA	0.28	0.38	0.33	0.42	0.33	0.19	0.07
BA	0.17	0.19	0.23	0.38	0.34	0.28	0.23
AB	0.18	0.23	0.28	0.40	0.33	0.27	0.20
Combined bi-quantic test	0.25	0.40	0.57	0.61	0.53	0.42	0.27
Quadrant association test	0.27	0.32	0.39	0.62	0.47	0.39	0.35
Single bi-quantic test			At 1% Critical Value				
BB	0.01	0.08	0.23	0.11	0.11	0.14	0.13
AA	0.12	0.16	0.12	0.24	0.18	0.07	0.01
BA	0.07	0.06	0.09	0.22	0.17	0.14	0.10
AB	0.10	0.09	0.14	0.24	0.16	0.10	0.07
Combined bi-quantic test	0.11	0.19	0.34	0.44	0.35	0.21	0.12
Quadrant association test	0.11	0.12	0.21	0.45	0.30	0.19	0.14

Notes to Table: We report rejection ratios by BQ test statistics on 3916 pairwise standardized returns for the 89 CDX companies from January 03, 1995 to December 03, 2004. The rejection ratio is the ratio of number of pairwise over critical value to the total number of pairwise. There are six test statistics here: four single bi-quantic tests (BB, AA, BA, and AB), combined bi-quantic test, and quadrant association test. The asymptotic distribution of these stationarity test statistics under the null hypothes that the C (quantile; quantile) is constant over time is Cramér von Mises.* (10% critical value for the combined test is not avalaible).

Table 2: Parameter Estimates for Dynamic Copulas, January 03, 1995 to December 03, 2004

DECO	ω	α	β				
Estimate	0.001	0.013	0.986				
Asymp. Std. Error	0.001	0.003	0.005				
Log Likelihood	21606.883						
DCC		α	β				
Estimate		0.009	0.986				
Asymp. Std. Error		0.003	0.006				
Log Likelihood	25400.770						
DECO Gaussian Copula	ω	α	β				
Estimate	0.002	0.019	0.979				
Asymp. Std. Error	0.001	0.004	0.009				
Log Likelihood	23498.474						
DCC Gaussian Copula		α	β				
Estimate		0.009	0.987				
Asymp. Std. Error		0.004	0.008				
Log Likelihood	24425.679						
DECO t Copula Constant DF	ω	α	β	DF			
Estimate	0.001	0.010	0.989	37.848			
Asymp. Std. Error	0.003	0.008	0.006	0.006			
Log Likelihood	25394.209						
DCC t Copula Constant DF		α	β	DoF			
Estimate		0.009	0.987	35.882			
Asymp. Std. Error		0.004	0.008	0.936			
Log Likelihood	27903.931						
DECO t Copula Dynamic DF	ω_corr	α_corr	β_corr	ω_dof	α_dof	β_dof	mean DF
Estimate	0.003	0.028	0.969	0.970	0.023	0.955	37.628
Asymp. Std. Error	0.001	0.001	0.002	0.003	0.004	0.000	
Log Likelihood	25419.824						
LR Test P-value against Constant DF	0.000						

	DECO Grouped t Copula			DCC Grouped t Copula		
	DF	Asymp. Std. Error	Log Likelihood	DF	Asymp. Std. Error	Log Likelihood
Industrial	18.842	1.004	4139.542	25.550	1.640	5098.564
Materials	7.612	0.396	2073.229	16.055	1.464	3156.086
Consumer Cyclical	25.234	1.281	6401.924	34.026	2.082	7401.211
Consumer Stable	26.869	2.143	2438.961	31.860	2.711	2897.557
Utilities	4.511	0.199	2069.089	13.906	1.451	2737.221
Energy	6.980	0.455	1550.100	16.039	1.873	2147.462
Communications and Technology	16.750	1.414	1591.035	20.884	2.018	1896.023
Financial	15.078	0.941	2986.970	21.948	1.679	3949.562

Notes to Table: We report parameter estimates for DECO/DCC and DECO/DCC copulas for the 89 CDX companies. The sample period for equity returns is from January 03, 1995 to December 03, 2004. The DCC and DCC copulas are estimated by maximizing m-profile subset composite likelihood (MSCL, contiguous pairs).

Table 3: Back-Testing Analysis for Dynamic Conditional Correlation, December 08, 2000 to December 03, 2004

		DECO						DCC					
Weight	**VaR**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**
Assuming Standardized Returns from Multi-Normal Distribution													
EW	10%	0.121	0.021	0.032	0.027	0.033	0.023	0.121	0.021	0.032	0.022	0.061	0.004
VW		0.134	0.001	0.001	0.146	0.332	0.000	0.116	0.057	0.099	0.035	0.601	0.030
EW	5%	0.069	0.009	0.009	0.000	0.020	0.036	0.071	0.005	0.004	0.000	0.030	0.031
VW		0.083	0.000	0.000	0.216	0.197	0.000	0.062	0.058	0.093	0.004	0.041	0.009
EW	1%	0.018	0.029	0.022	0.197	0.320	0.015	0.021	0.008	0.002	0.120	0.442	0.129
VW		0.030	0.000	0.000	0.009	0.011	0.029	0.019	0.019	0.011	0.073	0.048	0.152
EW	0.5%	0.016	0.003	0.000	0.048	0.247	0.282	0.013	0.013	0.003	0.024	0.544	0.177
VW		0.015	0.005	0.000	0.009	0.485	0.014	0.009	0.090	0.107	0.995	0.670	0.041
Assuming Standardized Returns from Multi-Student's t Distribution													
EW	10%	0.117	0.047	0.080	0.006	0.030	0.017	0.117	0.047	0.080	0.015	0.057	0.006
VW		0.135	0.001	0.000	0.072	0.363	0.000	0.116	0.057	0.099	0.018	0.601	0.069
EW	5%	0.066	0.021	0.027	0.000	0.031	0.039	0.070	0.007	0.006	0.000	0.025	0.045
VW		0.081	0.000	0.000	0.126	0.152	0.002	0.062	0.058	0.093	0.005	0.041	0.032
EW	1%	0.017	0.043	0.043	0.106	0.282	0.086	0.021	0.008	0.002	0.120	0.442	0.190
VW		0.026	0.001	0.000	0.011	0.029	0.038	0.017	0.043	0.043	0.475	0.282	0.162
EW	0.5%	0.013	0.013	0.003	0.004	0.544	0.315	0.013	0.013	0.003	0.024	0.544	0.266
VW		0.015	0.005	0.000	0.594	0.485	0.057	0.009	0.090	0.107	0.995	0.670	0.066
Assuming Portfolio Returns from Univariate Student's t Distribution													
EW	10%	0.128	0.004	0.004	0.043	0.062	0.957	0.132	0.001	0.001	0.060	0.115	0.817
VW		0.148	0.000	0.000	0.316	0.250	0.187	0.129	0.003	0.003	0.358	0.519	0.616
EW	5%	0.060	0.092	0.159	0.003	0.028	0.957	0.064	0.035	0.051	0.002	0.059	0.870
VW		0.084	0.000	0.000	0.480	0.222	0.343	0.061	0.073	0.122	0.001	0.034	0.417
EW	1%	0.011	0.381	0.754	0.001	0.605	0.981	0.012	0.281	0.538	0.007	0.574	0.979
VW		0.018	0.029	0.022	0.042	0.038	0.486	0.009	0.369	0.746	0.995	0.670	0.402
EW	0.5%	0.002	0.017	0.126	1.000	0.913	0.516	0.002	0.017	0.126	1.000	0.913	0.459
VW		0.006	0.341	0.664	0.999	0.771	0.202	0.003	0.124	0.333	1.000	0.877	0.193

Notes to Table: We report p-values for unconditional coverage tests, independence tests, and ES bootstrap tests. Hitting Ratio is the proportion of the realized loss returns in excess of VaR. MT is the simple means test. LRuc is the unconditional coverage likelihood ratio test. LB is the Ljung-Box autocorrelation test. LRcc is the independence likelihood ratio test. ES Bootstrap Test is based on the exceedance residuals which are the realized returns minus ES whenever violation occurs. The ex ante VaR and ES are from DCC and DECO for both equally-weighted portfolios and value-weighted portfolios updated annually from December 08, 2000 to December 03, 2004.

Table 4: Back-Testing Analysis for Constant Copula, December 08, 2000 to December 03, 2004

		Average Equal Dependence						Dependence Matrix					
Weight	**VaR**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**
							Gaussian Copula						
EW	10%	0.128	0.004	0.004	0.081	0.171	0.000	0.126	0.007	0.008	0.060	0.131	0.000
VW		0.149	0.000	0.000	0.086	0.374	0.000	0.120	0.026	0.040	0.105	0.602	0.023
EW	5%	0.077	0.001	0.000	0.000	0.085	0.002	0.078	0.000	0.000	0.000	0.044	0.005
VW		0.097	0.000	0.000	0.049	0.313	0.001	0.069	0.009	0.009	0.044	0.054	0.078
EW	1%	0.022	0.005	0.001	0.200	0.485	0.002	0.022	0.005	0.001	0.200	0.485	0.013
VW		0.031	0.000	0.000	0.154	0.078	0.005	0.020	0.012	0.005	0.678	0.400	0.412
EW	0.5%	0.017	0.002	0.000	0.182	0.282	0.092	0.014	0.008	0.001	0.017	0.183	0.090
VW		0.021	0.000	0.000	0.718	0.442	0.073	0.005	0.500	1.000	1.000	0.806	0.000
							Student's T Copula						
EW	10%	0.126	0.007	0.008	0.060	0.131	0.000	0.124	0.011	0.014	0.056	0.099	0.000
VW		0.151	0.000	0.000	0.117	0.435	0.000	0.121	0.021	0.032	0.144	0.586	0.038
EW	5%	0.078	0.000	0.000	0.000	0.044	0.001	0.078	0.000	0.000	0.000	0.044	0.002
VW		0.094	0.000	0.000	0.028	0.227	0.001	0.070	0.007	0.006	0.065	0.064	0.119
EW	1%	0.022	0.005	0.001	0.200	0.485	0.000	0.022	0.005	0.001	0.200	0.485	0.000
VW		0.033	0.000	0.000	0.035	0.110	0.001	0.022	0.005	0.001	0.214	0.088	0.381
EW	0.5%	0.019	0.001	0.000	0.405	0.359	0.005	0.016	0.003	0.000	0.100	0.247	0.049
VW		0.022	0.000	0.000	0.729	0.485	0.009	0.008	0.144	0.216	0.997	0.703	0.140

Notes to Table: We report p-values for unconditional coverage tests, independence tests, and ES bootstrap tests. Hitting Ratio is the proportion of the realized loss returns in excess of VaR. MT is the simple means test. LRuc is the unconditional coverage likelihood ratio test. LB is the Ljung-Box autocorrelation test. LRcc is the independence likelihood ratio test. ES Bootstrap Test is based on the exceedance residuals which are the realized returns minus ES whenever violation occurs. The ex ante VaR and ES are from constant copulas for both equally-weighted portfolios and value-weighted portfolios updated annually from December 08, 2000 to December 03, 2004.

Table 5: Back-Testing Analysis for Dynamic Gaussian Copula, December 08, 2000 to December 03, 2004

		DECO Gaussian Copula						DCC Gaussian Copula					
Weight	**VaR**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**
EW	10%	0.100	0.500	1.000	0.002	0.009	0.428	0.107	0.237	0.465	0.002	0.017	0.049
VW		0.114	0.082	0.148	0.034	0.562	0.036	0.111	0.134	0.254	0.028	0.596	0.257
EW	5%	0.051	0.443	0.885	0.004	0.014	0.433	0.060	0.092	0.159	0.000	0.009	0.166
VW		0.064	0.035	0.051	0.003	0.021	0.043	0.055	0.244	0.475	0.003	0.010	0.081
EW	1%	0.008	0.239	0.510	0.000	0.703	0.337	0.013	0.201	0.362	0.024	0.544	0.320
VW		0.014	0.141	0.231	0.518	0.514	0.120	0.014	0.141	0.231	0.518	0.514	0.343
EW	0.5%	0.002	0.017	0.126	1.000	0.913	0.134	0.005	0.500	1.000	0.000	0.806	0.217
VW		0.010	0.056	0.049	0.121	0.637	0.311	0.006	0.341	0.664	0.999	0.771	0.211
		Apply EVT to Simulated Portfolio Returns that Fall in Each 10% Tail.											
EW	10%	0.092	0.191	0.393	0.019	0.179	0.450	0.102	0.417	0.834	0.003	0.013	0.072
VW		0.111	0.134	0.254	0.259	0.475	0.074	0.103	0.378	0.753	0.104	0.628	0.237
EW	5%	0.045	0.223	0.461	0.014	0.058	0.418	0.058	0.140	0.257	0.000	0.019	0.225
VW		0.057	0.170	0.320	0.002	0.004	0.012	0.050	0.500	1.000	0.002	0.003	0.063
EW	1%	0.008	0.239	0.510	0.000	0.703	0.324	0.013	0.201	0.362	0.024	0.544	0.289
VW		0.014	0.141	0.231	0.518	0.514	0.164	0.012	0.281	0.538	0.980	0.574	0.220
EW	0.5%	0.002	0.017	0.126	1.000	0.913	0.156	0.005	0.500	1.000	0.000	0.806	0.180
VW		0.007	0.224	0.398	0.998	0.737	0.202	0.004	0.308	0.642	1.000	0.841	0.046

Notes to Table: We report p-values for unconditional coverage tests, independence tests, and ES bootstrap tests. Hitting Ratio is the proportion of the realized loss returns in excess of VaR. MT is the simple means test. LRuc is the unconditional coverage likelihood ratio test. LB is the Ljung-Box autocorrelation test. LRcc is the independence likelihood ratio test. ES Bootstrap Test is based on the exceedance residuals which are the realized returns minus ES whenever violation occurs. The ex ante VaR and ES are from DCC/DECO Gaussian copulas for both equally-weighted portfolio and value-weighted portfolios updated annually from December 08, 2000 to December 03, 2004.

Table 6: VaR Back-Testing Analysis for Dynamic t-Copula, December 08, 2000 to December 03, 2004

								Apply EVT to Simulated Portfolio Returns that Fall in Each 10% Tail					
Weight	**VaR**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**
DECO t Copula													
EW	10%	0.101	0.458	0.916	0.002	0.011	0.301	0.093	0.223	0.456	0.005	0.110	0.313
VW		0.117	0.047	0.080	0.072	0.526	0.042	0.111	0.134	0.254	0.259	0.475	0.069
EW	5%	0.052	0.388	0.773	0.004	0.018	0.304	0.045	0.223	0.461	0.014	0.058	0.216
VW		0.061	0.073	0.122	0.002	0.011	0.012	0.057	0.170	0.320	0.002	0.004	0.011
EW	1%	0.012	0.281	0.538	0.007	0.574	0.366	0.011	0.381	0.754	0.001	0.605	0.326
VW		0.014	0.141	0.231	0.518	0.514	0.038	0.014	0.141	0.231	0.518	0.514	0.126
EW	0.5%	0.003	0.124	0.333	0.000	0.877	0.123	0.003	0.124	0.333	0.000	0.877	0.136
VW		0.010	0.056	0.049	0.121	0.637	0.159	0.008	0.144	0.216	0.997	0.703	0.176
DCC t Copula													
EW	10%	0.107	0.237	0.465	0.001	0.017	0.020	0.102	0.417	0.834	0.003	0.013	0.021
VW		0.108	0.208	0.405	0.117	0.521	0.131	0.103	0.378	0.753	0.104	0.628	0.149
EW	5%	0.064	0.035	0.051	0.000	0.021	0.128	0.060	0.092	0.159	0.000	0.028	0.128
VW		0.057	0.170	0.320	0.001	0.015	0.117	0.052	0.388	0.773	0.003	0.004	0.068
EW	1%	0.017	0.043	0.043	0.106	0.282	0.326	0.016	0.065	0.079	0.048	0.247	0.307
VW		0.014	0.141	0.231	0.518	0.514	0.279	0.011	0.381	0.754	0.987	0.605	0.183
EW	0.5%	0.010	0.056	0.049	0.000	0.637	0.416	0.007	0.224	0.398	0.002	0.737	0.255
VW		0.006	0.341	0.664	0.999	0.771	0.128	0.006	0.341	0.664	0.999	0.771	0.187
DECO t Copula with Dynamic DF													
EW	10%	0.101	0.458	0.916	0.002	0.011	0.296	0.093	0.223	0.456	0.005	0.110	0.307
VW		0.117	0.047	0.080	0.072	0.526	0.042	0.111	0.134	0.254	0.259	0.475	0.069
EW	5%	0.052	0.388	0.773	0.004	0.018	0.300	0.045	0.223	0.461	0.014	0.058	0.221
VW		0.060	0.092	0.159	0.003	0.009	0.007	0.057	0.170	0.320	0.002	0.004	0.011
EW	1%	0.012	0.281	0.538	0.007	0.574	0.368	0.011	0.381	0.754	0.001	0.605	0.320
VW		0.016	0.065	0.079	0.009	0.247	0.104	0.014	0.141	0.231	0.518	0.514	0.125
EW	0.5%	0.003	0.124	0.333	0.000	0.877	0.111	0.003	0.124	0.333	0.000	0.877	0.117
VW		0.009	0.090	0.107	0.052	0.670	0.119	0.008	0.144	0.216	0.997	0.703	0.181

Notes to Table: We report p-values for unconditional coverage tests, independence tests, and ES bootstrap tests. Hitting Ratio is the proportion of the realized loss returns in excess of VaR. MT is the simple means test. LRuc is the unconditional coverage likelihood ratio test. LB is the Ljung-Box autocorrelation test. LRcc is the independence likelihood ratio test. ES Bootstrap Test is based on the exceedance residuals which are the realized returns minus ES whenever violation occurs. The ex ante VaR and ES are from DCC/DECO t-copula for both equally-weighted portfolios and value-weighted portfolios updated annually from December 08, 2000 to December 03, 2004.

Table 7: VaR Back-Testing Analysis for Grouped t-Copula, December 08, 2000 to December 03, 2004

		DECO Grouped t Copula						DCC Grouped t Copula					
Weight	**VaR**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**	**Hitting Ratio**	**MT P-value**	**LRuc P-value**	**LB P-value**	**LRcc P-value**	**ES Bootstrap P-value**
EW	10%	0.102	0.417	0.834	0.003	0.013	0.286	0.109	0.181	0.349	0.001	0.012	0.041
VW		0.117	0.047	0.080	0.072	0.526	0.032	0.111	0.134	0.254	0.028	0.596	0.190
EW	5%	0.056	0.205	0.393	0.005	0.040	0.504	0.064	0.035	0.051	0.000	0.021	0.193
VW		0.066	0.021	0.027	0.010	0.031	0.041	0.056	0.205	0.393	0.005	0.012	0.076
EW	1%	0.009	0.369	0.746	0.000	0.670	0.343	0.016	0.065	0.079	0.048	0.247	0.339
VW		0.013	0.201	0.362	0.427	0.544	0.072	0.014	0.141	0.231	0.518	0.514	0.275
EW	0.5%	0.002	0.017	0.126	1.000	0.913	0.112	0.007	0.224	0.398	0.002	0.737	0.275
VW		0.009	0.090	0.107	0.995	0.670	0.181	0.006	0.341	0.664	0.999	0.771	0.125
Apply EVT to Simulated Portfolio Returns that Fall in Each 10% Tail.													
EW	10%	0.095	0.295	0.595	0.006	0.037	0.358	0.104	0.339	0.675	0.001	0.009	0.062
VW		0.113	0.097	0.179	0.069	0.535	0.070	0.104	0.339	0.675	0.103	0.541	0.239
EW	5%	0.047	0.327	0.660	0.028	0.081	0.345	0.058	0.140	0.257	0.000	0.019	0.143
VW		0.057	0.170	0.320	0.002	0.004	0.006	0.051	0.443	0.885	0.001	0.003	0.067
EW	1%	0.009	0.369	0.746	0.000	0.670	0.359	0.015	0.097	0.139	0.017	0.214	0.324
VW		0.015	0.097	0.139	0.009	0.485	0.178	0.009	0.369	0.746	0.995	0.670	0.038
EW	0.5%	0.002	0.017	0.126	1.000	0.913	0.128	0.005	0.500	1.000	0.000	0.806	0.154
VW		0.009	0.090	0.107	0.995	0.670	0.200	0.006	0.341	0.664	0.999	0.771	0.147

Notes to Table: We report p-values for unconditional coverage tests, independence tests, and ES bootstrap tests. Hitting Ratio is the proportion of the realized loss returns in excess of VaR. MT is the simple means test. LRuc is the unconditional coverage likelihood ratio test. LB is the Ljung-Box autocorrelation test. LRcc is the independence likelihood ratio test. ES Bootstrap Test is based on the exceedance residuals which are the realized returns minus ES whenever violation occurs. The ex ante VaR and ES are from DCC/DECO grouped t-copula for both equally-weighted portfolios and value-weighted portfolios updated annually from December 08, 2000 to December 03, 2004.

Table 8: Pair-wise Tests for Out-of-sample Performance of Copulas, December 08, 2000 to December 03, 2004

	DECO Gaussian	DECO t	DECO Dynamic t	DCC Gaussian	DCC t
Log likelihood	13030.69	14244.36	14246.39	17973.20	18694.58
		Q Test P-Value			
DECO t	0.00				
DECO Dynamic t	0.00	0.41			
DCC Gaussian	0.00	0.00	0.00		
DCC t	0.00	0.00	0.00	0.00	

	Log likelihood		Q Test	Log likelihood		Q Test
	DECO Grouped t	DECO t	DECO P-Value	DCC Grouped t	DCC t	DCC P-Value
Industrial	2434.94	2379.98	0.02	3203.82	3173.90	0.10
Materials	1203.43	1082.56	0.00	1900.20	1867.26	0.00
Consumer Cyclical	3807.15	3775.63	0.01	4613.03	4599.08	0.01
Consumer Stable	1307.67	1296.28	0.04	1519.11	1514.47	0.11
Utilities	899.91	844.00	0.13	1171.75	1161.69	0.22
Energy	879.41	804.37	0.00	1187.12	1167.27	0.03
Communications and Technology	974.95	947.97	0.01	1172.48	1155.05	0.01
Financial	1646.76	1601.15	0.01	2068.59	2046.89	0.01
Total	13154.21	12731.94	0.00	16836.10	16685.61	0.00

Notes to Table: We report the P-values of the Q test statistic of the null hypothesis of equal predictive accuracy of two alternative copula specifications. P-values indicate better performance of copula with the larger log likelihood. The number of observations in the out-of-sample period is 1000 from December 08, 2000 to December 03, 2004.

Table 9: Statistics of Realized Optimal Portfolio Returns, December 08, 2000 to December 03, 2004

	DECO Gaussian Copula		DECO t Copula
	MinVar Weighted	MinES Weighted	MinES Weighted
Mean	3.8552E-04	4.2280E-04	3.7531E-04
STD	7.3910E-03	7.2514E-03	7.3122E-03
Skewness	-0.0452	-0.0727	-0.1039
Kurtosis	5.2997	5.4297	5.3313
Autocorr	0.1002	0.0798	0.0835
Lbqtest pValue	0.0261	0.0391	0.1009
Sharpe Ratio	0.0522	0.0583	0.0513
VaR (10%)	8.0483E-03	8.1133E-03	8.1726E-03
ES (10%)	1.3263E-02	1.2896E-02	1.3058E-02

	DCC Gaussian Copula		DCC t Copula
	MinVar Weighted	MinES Weighted	MinES Weighted
Mean	4.5498E-04	4.4826E-04	4.1925E-04
STD	6.8353E-03	6.8483E-03	6.9053E-03
Skewness	-0.3149	-0.2843	-0.3196
Kurtosis	5.5860	5.3284	5.4421
Autocorr	0.1412	0.1307	0.1283
Lbqtest pValue	0.0001	0.0001	0.0006
Sharpe Ratio	0.0666	0.0655	0.0607
VaR (10%)	7.7227E-03	7.7030E-03	7.9233E-03
ES (10%)	1.2457E-02	1.2315E-02	1.2450E-02

Notes to Table: We report the statistics for the realized optimal portfolios by both Mean-Variance and ES criteria under no-shorting from December 08, 2000 to December 03, 2004.

Table 10: Back-Testing Analysis for the optimal portfolios, December 08, 2000 to December 03, 2004

		DECO Copula						DECO Copula at DF = 15					
Weight	VaR	Hitting Ratio	MT P-value	LRuc P-value	LB P-value	LRcc P-value	ES Bootstrap P-value	Hitting Ratio	MT P-value	LRuc P-value	LB P-value	LRcc P-value	ES Bootstrap P-value
MinVar (Gaussian)		0.093	0.223	0.456	0.000	0.000	0.027	0.098	0.416	0.833	0.000	0.000	0.135
MinES (Gaussian)	10%	0.085	0.045	0.105	0.000	0.001	0.001	0.089	0.111	0.238	0.000	0.000	0.227
MinES (Student's t)		0.090	0.135	0.284	0.003	0.002	0.006	0.093	0.223	0.456	0.000	0.000	0.266
MinVar (Gaussian)		0.052	0.388	0.773	0.000	0.000	0.017	0.053	0.336	0.666	0.000	0.000	0.214
MinES (Gaussian)	5%	0.054	0.288	0.566	0.000	0.000	0.056	0.053	0.336	0.666	0.000	0.000	0.654
MinES (Student's t)		0.051	0.443	0.885	0.000	0.000	0.006	0.053	0.336	0.666	0.000	0.000	0.521
MinVar (Gaussian)		0.016	0.065	0.079	0.652	0.247	0.413	0.014	0.141	0.231	0.961	0.514	0.353
MinES (Gaussian)	1%	0.011	0.381	0.754	0.987	0.605	0.083	0.010	0.500	1.000	0.992	0.637	0.265
MinES (Student's t)		0.016	0.065	0.079	0.652	0.247	0.135	0.010	0.500	1.000	0.992	0.637	0.198
MinVar (Gaussian)		0.006	0.341	0.664	0.999	0.771	0.209	0.005	0.500	1.000	1.000	0.806	0.147
MinES (Gaussian)	0.5%	0.007	0.224	0.398	0.998	0.737	0.137	0.003	0.124	0.333	1.000	0.877	0.007
MinES (Student's t)		0.008	0.144	0.216	0.997	0.703	0.137	0.003	0.124	0.333	1.000	0.877	0.008
		DCC Copula						DCC Copula at DF = 7					
MinVar (Gaussian)		0.103	0.378	0.753	0.001	0.001	0.000	0.107	0.237	0.465	0.000	0.000	0.075
MinES (Gaussian)	10%	0.105	0.303	0.601	0.000	0.000	0.000	0.111	0.134	0.254	0.000	0.000	0.160
MinES (Student's t)		0.110	0.156	0.299	0.000	0.001	0.001	0.105	0.303	0.601	0.000	0.000	0.042
MinVar (Gaussian)		0.062	0.058	0.093	0.000	0.000	0.001	0.061	0.073	0.122	0.000	0.000	0.225
MinES (Gaussian)	5%	0.062	0.058	0.093	0.000	0.000	0.000	0.060	0.092	0.159	0.000	0.002	0.237
MinES (Student's t)		0.063	0.045	0.069	0.000	0.000	0.001	0.058	0.140	0.257	0.000	0.000	0.173
MinVar (Gaussian)		0.025	0.001	0.000	0.000	0.000	0.353	0.013	0.201	0.362	0.000	0.009	0.730
MinES (Gaussian)	1%	0.024	0.002	0.000	0.000	0.018	0.088	0.012	0.281	0.538	0.053	0.128	0.290
MinES (Student's t)		0.021	0.008	0.002	0.000	0.008	0.006	0.011	0.381	0.754	0.217	0.105	0.140
MinVar (Gaussian)		0.014	0.008	0.001	0.000	0.001	0.381	0.005	0.500	1.000	0.000	0.016	0.469
MinES (Gaussian)	0.5%	0.015	0.005	0.000	0.000	0.001	0.105	0.007	0.224	0.398	0.002	0.036	0.327
MinES (Student's t)		0.015	0.005	0.000	0.000	0.001	0.057	0.007	0.224	0.398	0.002	0.036	0.300

Notes to Table: We report p-values for unconditional coverage tests, independence tests, and ES bootstrap tests. Hitting Ratio is the proportion of the realized loss returns in excess of VaR. MT is the simple means test. LRuc is the unconditional coverage likelihood ratio test. LB is the Ljung-Box autocorrelation test. LRcc is the independence likelihood ratio test. ES Bootstrap Test is based on the exceedance residuals which are the realized returns minus ES whenever violation occurs. The ex ante VaR and ES are estimated by DECO/DCC copulas and Student's t at a given degrees of freedom with covariance from DECO/DCC copula for the 89 CDX companies from December 08, 2000 to December 03, 2004. The optimal portfolios are constructed by Mean-Variance and ES criteria under no-shorting.

Table 11: The List of 89 CDX Firms

Company Name	Sector	Average Rating	Company Name	Sector	Average Rating
Honeywell International Inc	Industrial	A	Sherwin Williams Co	Consumer Cyclical	A
Fortune Brands Inc	Consumer Stable	BBB	Weyerhaeuser Co	Materials	BBB
Du Pont E I De Nemours & Co	Materials	A	Computer Sciences Corp	Communications and Technology	BBB
General Electric Co	Financial	AAA	Mcdonalds Corp	Consumer Cyclical	A
Goodrich Corp	Industrial	BBB	Marsh & Mclennan Cos Inc	Financial	BBB
Ingersoll Rand Co Ltd	Industrial	A	New York Times Co	Communications and Technology	BB
International Business Machs Cor	Communications and Technology	A	Gannett Inc	Communications and Technology	BBB
Altria Group Inc	Consumer Stable	BBB	Union Pacific Corp	Industrial	BBB
Conocophillips	Energy	A	Target Corp	Consumer Cyclical	A
Amgen Inc	Consumer Stable	A	Wal Mart Stores Inc	Consumer Cyclical	AA
Wyeth	Consumer Stable	A	Conagra Inc	Consumer Cyclical	BBB
Kroger Company	Consumer Stable	BBB	Nordstrom Inc	Consumer Cyclical	A
C V S Caremark Corp	Consumer Cyclical	BBB	Southwest Airlines Co	Consumer Cyclical	BBB
General Mills Inc	Consumer Stable	BBB	American Express Co	Financial	A
Penney J C Co Inc	Consumer Cyclical	BBB	Chubb Corp	Financial	A
Caterpillar Inc	Consumer Cyclical	A	Centurytel Inc	Communications and Technology	BBB
Deere & Co	Consumer Cyclical	A	Newell Rubbermaid Inc	Industrial	BBB
Bristol Myers Squibb Co	Consumer Stable	A	C S X Corp	Industrial	BBB
Boeing Co	Financial	A	C I G N A Corp	Financial	BBB
Black & Decker Corp	Industrial	BBB	Limited Brands Inc	Consumer Cyclical	BB
Dow Chemical Co	Materials	BBB	Norfolk Southern Corp	Industrial	BBB
Lockheed Martin Corp	Industrial	A	Dominion Resources Inc Va New	Utilities	BBB
Cardinal Health Inc	Consumer Cyclical	BBB	A T & T Inc	Communications and Technology	A
International Paper Co	Materials	BBB	Home Depot Inc	Consumer Cyclical	BBB
Motorola Inc	Industrial	BBB	M D C Holdings Inc	Consumer Cyclical	BBB
Sara Lee Corp	Consumer Stable	BBB	American International Group Inc	Financial	A
Firstenergy Corp	Utilities	BBB	Toll Brothers Inc	Consumer Cyclical	BBB
Progress Energy Inc	Utilities	BBB	Anadarko Petroleum Corp	Energy	BBB
Halliburton Company	Energy	BBB	Barrick Gold Corp	Materials	BBB
American Electric Power Co Inc	Utilities	BBB	Carnival Corp	Industrial	A
Constellation Energy Group Inc	Utilities	BBB	Staples Inc	Consumer Cyclical	BBB
Alcoa Inc	Materials	BBB	Istar Financial Inc	Financial	BBB
Northrop Grumman Corp	Industrial	BBB	Safeway Inc	Consumer Stable	BBB
Raytheon Co	Industrial	A	Autozone Inc	Consumer Cyclical	BBB
Campbell Soup Co	Consumer Stable	A	Macys Inc	Consumer Cyclical	BBB
Whirlpool Corp	Consumer Cyclical	BBB	Mohawk Industries Inc	Consumer Stable	BBB
Disney Walt Co	Consumer Cyclical	A	Kohls Corp	Consumer Cyclical	BBB
Loews Corp	Consumer Stable	A	Ace Ltd	Financial	A
Ryder Systems Inc	Industrial	A	X T O Energy Inc	Energy	BBB
Hewlett Packard Co	Communications and Technology	A	Transocean Inc New	Energy	BBB
Baxter International Inc	Consumer Stable	A	Allstate Corp	Financial	A
Xerox Corp	Consumer Cyclical	BBB	Universal Health Services Inc	Consumer Stable	BBB
Arrow Electronics Inc	Industrial	BBB	Eastman Chemical Co	Materials	BBB
Omnicom Group Inc	Communications and Technology	A	Simon Property Group Inc New	Financial	A
Masco Corp	Industrial	BBB			