1 INTRODUCTION

In 1920, Ludwig Prandtl published an analytical solution for the bearing capacity of a soil under a strip load, causing kinematic failure of the weightless infinite half-space underneath. The strength of the half-space is given by the angle of internal friction, \( \phi \), and the cohesion, \( c \). The original drawing of the failure mechanism proposed by Prandtl can be seen in Figure 1.

![Figure 1. The Prandtl-wedge failure mechanism (Original drawing by Prandtl).](image)

The lines in the sliding soil part on the left indicate the directions of the maximum and minimum principal stresses, while the lines in the sliding soil part on the right, indicate the sliding lines with a direction of \( \alpha = 45^\circ - \frac{1}{2} \phi \) in comparison to the maximum principal stress. Prandtl subdivided the sliding soil part into three zones:

1. **Zone 1:** A triangular zone below the strip load. Since there is no friction on the ground surface, the directions of the principal stresses are horizontal and vertical; the largest principal stress is in the vertical direction.

2. **Zone 2:** A wedge with the shape of a logarithmic spiral, in which the principal stresses rotate through \( \frac{1}{2} \pi \) radians, or 90 degrees, from Zone 1 to Zone 3. The pitch of the sliding surface of the logarithmic spiral equals the angle of internal friction; \( \phi \), creating a smooth transition between Zone 1 and Zone 3.

3. **Zone 3:** A triangular zone adjacent to the strip load. Since there is no friction on the surface of the ground, the directions of principal stress are horizontal and vertical; the largest principal stress is in the horizontal direction.

The solution of Prandtl was extended by Hans J. Reissner (1924) with a surrounding surcharge, \( q \), and was based on the same failure mechanism. Albert S. Keverling Buisman (1940) and Karl Terzaghi (1943) extended the Prandtl-Reissner formula for the soil weight, \( \gamma \). It was Terzaghi who first wrote the bearing capacity with three separate bearing capacity factors for the cohesion, surcharge and soil weight. George G. Meyerhof (1953) was the first to propose equations for inclined loads, based on his own laboratory experiments. Meyerhof was also the first in 1963 to write the formula for the (vertical) bearing capacity \( p_v \) with bearing capacity factors (\( N \)), inclination factors (\( i \)) and shape factors (\( s \)), for the three independent bearing components; cohesion (\( c \)), surcharge (\( q \)) and soil weight (\( \gamma \)), in a way it was adopted by Jørgen A. Brinch Hansen (1970) and it is still used nowadays:

\[
p_v = i_s s c N_c + i_s s q N_q + i_s s \gamma \frac{1}{2} \gamma B N_r.
\]

The bearing capacity of shallow foundations on slopes

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ABSTRACT: In 1920 Prandtl published an analytical solution for the bearing capacity of a soil under a strip load. Over the years, extensions have been made for a surrounding surcharge, the soil weight, the shape of the footing, the inclination of the load, and also for a slope. In order to check the current extensions of a loaded strip footing next to a slope, many finite element calculations have been made, showing that these extensions are often inaccurate. Therefore new factors have been proposed, which are for both the soil-weight and the surcharge slope bearing capacity, based on the numerical calculations, and for the cohesion slope bearing capacity, also on an analytical solution.
Prandtl (1920) solved the cohesion bearing capacity factor:

\[ N_c = (K_p \cdot e^\frac{\pi \tan \phi}{2} - 1) \cot \phi \quad \text{with:} \quad K_p = \frac{1 + \sin \phi}{1 - \sin \phi} \quad (2) \]

Reissner (1924) solved the surcharge bearing capacity factor with the equilibrium of moments of Zone 2:

\[ N_q = K_p \cdot \left( \frac{r_1}{r_2} \right)^2 = K_p \cdot e^{\pi \tan \phi} \quad \text{with:} \quad r_2 = r_1 \cdot e^{\frac{\pi \tan \phi}{2}} \quad (3) \]

Keverling Buisman (1940), Terzaghi (1943), Caquot and Kérisel (1953, 1966), Meyerhof (1951; 1953; 1963; 1965), Brinch Hansen (1970), Vesic (1973, 1975), and Chen (1975) subsequently proposed different equations for the soil-weight bearing capacity factor \( N_b \). Therefore the following equations for the soil-weight bearing capacity factor can be found in the literature:

\[ N_b = (K_p \cdot e^{\pi \tan \phi} - 1) \tan (1.4 \phi) \quad \text{(Meyerhof)}, \]
\[ N_b = 1.5 \left( K_p \cdot e^{\pi \tan \phi} - 1 \right) \tan \phi \quad \text{(Brinch Hansen)}, \]
\[ N_b = 2 \left( K_p \cdot e^{\pi \tan \phi} + 1 \right) \tan \phi \quad \text{(Vesic)}, \]
\[ N_b = 2 \left( K_p \cdot e^{\pi \tan \phi} - 1 \right) \tan \phi \quad \text{(Chen)}. \]

The problem with all these solutions is that they are all based on associated soil (\( \psi = \phi \)). Loukidis et al. (2008) noticed that non-dilatant (non-associated) soil is 15% - 30% weaker than associated soil, and has a rougher failure pattern. Van Baars (2015, 2016a, 2016b) confirmed these results with his numerical calculations and showed that, for non-dilatant soil, the following lower factors describe better the bearing capacity:

\[ N_q = \cos^2 \phi \cdot K_p \cdot e^{\pi \tan \phi} \quad (5) \]
\[ N_b = (N_q - 1) \cot \phi \quad \text{with:} \quad N_b = \cos^2 \phi \cdot K_p \cdot e^{\pi \tan \phi} \quad (6) \]
\[ N_b = 4 \tan \phi \cdot \left( e^{\pi \tan \phi} - 1 \right) \quad (7) \]

The difference between the analytical solution and the numerical results has been explained by Knudsen and Mortensen (2016): The higher the friction angle, the wider the logarithmic spiral of the Prandtl wedge and the more the stresses reduce in this wedge during failure. So, the analytical formulas are only kinematically admissible for an associated flow behaviour (\( \psi = \phi \)), which is completely unrealistic for natural soils. This means for higher friction angles as well that, a calculation of the bearing capacity of a footing based on the analytical solutions (Equations 2-3), is also unrealistic.

Therefore, in this study, the bearing capacity factors and the slope factors will be calculated with the software Plaxis 2D for a bi-linear constitutive Mohr-Coulomb \((c, \phi)\) soil model without hardening, softening, or volume change during failure (so the dilatancy angle \( \psi = 0 \)).

2 MEYERHOF & VESIC

Shallow foundations also exist in or near slopes, for example the foundation of a house or a bridge (Figure 2). Meyerhof was in 1957 the first to publish about the bearing capacity of foundations on a slope. He wrote: “Foundations are sometimes built on sloping sites or near the top edge of a slope... When a foundation located on the face of a slope is loaded to failure, the zones of plastic flow in the soil on the side of the slope are smaller than those of a similar foundation on level ground and the ultimate bearing capacity is correspondingly reduced”.

![Figure 2. Footing of a house and bridge near a slope.](image-url)
• the figures are not explained and cannot be verified,
• the figures are only for purely cohesive or purely frictional soil,
• the important angle $\alpha$ (see line EA in the figure on the left) is never solved,
• the slope bearing capacity does not go to “0” for $\beta = \phi$, and
• the reduced bearing capacity factors of Meyerhof are too high according to the results of Finite Element Model (FEM load controlled) calculations, made in this article (see the added points in Figure 3 below).

Brinch Hansen (1970) also worked on the influence of a slope. Vesic (1975) combined the work of Meyerhof and Brinch Hansen and proposed the following bearing capacity equation:

$$p = \lambda_c N_c + \lambda_q q N_q + \lambda_r \gamma BN_r,$$

(8)

with the following slope factors:

$$\lambda_c = \frac{N_c \lambda_q - 1}{N_q - 1} \quad (\phi > 0),$$

$$\lambda_q = 1 - \frac{2\beta}{\pi + 2} \quad (\phi = 0),$$

$$\lambda_r = \lambda_r = (1 - \tan \beta)^2.$$  

(9)

The angles in these equations are in radians.

It is remarkable, if not to say impossible, that these slope factors do not depend on the friction angle $\phi$, and that the surcharge slope factor $\lambda_q$ and the soil-weight slope factor $\lambda_c$ are identical.

Another mistake is that the cohesion slope factor $N_c$ is solved based on the assumption that Equation 6 about the relation between the cohesion bearing capacity $N_c$ and the surcharge bearing capacity $N_q$, is also valid for inclined loading, and also for loading near a slope ($\lambda_c N_c = (\lambda_q N_q - 1)\cot \phi$). This assumption was published first by De Beer and Ladanyi (1961). Vesic (1975) calls this “the theorem of correspondence”, and Bolton (1975) calls this “the usual trick”.

The relation between $N_c$ and $N_q$ in Equation 6 is coincidentally valid for vertical ultimate loads without a slope ($N_c = (N_q - 1)\cot \phi$), but the assumption that this is also the case for inclined loading and loading near a slope, is not correct, according to the results of the numerical calculations, and also according to the analytical solution given later in this paper.

This indicates that not only the inclination factors, but also the slope factors proposed by Vesic, are incorrect and should not be used.

3 MODERN RESEARCH & GERMAN NORMS

Over the years quite some people have published about the bearing capacity of footings on a slope, but that was mostly limited to, or purely cohesive slopes (Azzouz and Baligh, 1983; Graham et al., 1988, Georgiadis, 2010, Shiau et al (2011) or purely non-cohesive slopes (Grahams et al., 1988), in a geotechnical centrifuge (Shields et al. 1990), or dedicated to even more complex cases like seismicity (Kumar and Rao, 2003; Yamamoto, 2010), reinforced soil (Alamshahi and Hataf, 2009; Choudhary et al., 2010) or 3D load cases near slopes (Michalowski, 1989; De Butan and Garnier, 1998), while the more simple non-seismic, non-reinforced, 2D situation is still not fully understood.

Figure 3. Above: failure mechanism of a footing in a slope. Below: reduced bearing capacity factors (according to Meyerhof, 1957).
Chakraborty and Kumar (2013) were one of the first to make a more general study, but unfortunately only used, as most researchers, the lower bound finite element limit analysis with a non-linear optimization. They also did not present slope correction factors. The same applies to Leshchinsky (2015), who used an upper-bound limit state plasticity failure discretisation scheme.

The currently most used slope correction factors, are the following slope correction factors mentioned in the German design norm (in fact the German Annex to Eurocode 7 “Geotechnical Engineering”):

\[
\lambda_c = \frac{N_q e^{-\alpha}}{N_q - 1} \quad (\phi > 0),
\]

\[
\lambda_s = 1 - 0.4 \tan \beta \quad (\phi = 0),
\]

\[
\lambda_q = (1 - \tan \beta)^{1.9},
\]

\[
\lambda_y = (1 - 0.5 \tan \beta)^6,
\]

in which: \( \alpha = 0.0349 \cdot \beta \cdot \tan \phi \).

The angles in these equations are in degrees and to avoid slope failure: \( \beta \leq \phi \).

There is no reference or any background information in the German design norm about these factors, which is a major problem. It is also remarkable, for these slope correction factors in the German norm, if not to say impossible, that the surcharge slope factor and also the soil-weight slope factor do not depend on the friction angle.

Because of these problems, the bearing capacity near slopes has been studied with the well-established and validated Finite Element Model Plaxis. First load controlled calculations have been made, and second, comparisons have been made between these Finite Element calculations and the results of the German design norm, the results of Bishop slip circle calculations (with the program “GEO5” from “Fine Civil Engineering Software”) and, for the cohesion slope factor \( \lambda_c \), also the results of the analytical solution proposed in this article.

4 SLOPE FACTORS

4.1 Cohesion slope factor \( \lambda_c \)

For two different friction angles \( \phi = 0^\circ, 30^\circ \) and four different slope angles \( \beta = 0^\circ, 10^\circ, 20^\circ, 30^\circ \), the failure mechanism for a cohesive \( (c = 10 \text{ kPa}) \), weightless \( (\gamma' = 0 \text{ kN/m}^3) \) soil has been calculated with numerical calculations (FEM) and compared with the Prandtl failure mechanism, see Figure 4.
This figure shows that a Prandtl-wedge with a reduced Zone 2 (the logarithmic spiral wedge) describes in general the failure mechanism.

Because of this, it is also possible to derive an analytical solution for the cohesion slope factor, in the same way as the derivation of the cohesion inclination factor \( i_c \) (see Van Baars, 2014):

\[
\lambda_c = \cos \beta \left( e^{-2\beta \tan \phi} - \frac{2\beta}{2 + \pi} e^{-\pi \tan \phi} \right) \quad \beta \leq \phi
\]  
\( (11) \)

The results of this analytical solution, the German design norm and the Bishop’s slip circle method have been plotted in Figure 5, together with the results from the Finite Element calculations. Figure 5 shows that the Bishop calculations are only correct for a zero friction angle. The analytical solution functions very well. The German norm would function just as good, if not the Prandtl solution for large dilatancy (Equation 2), but the solution for zero dilatancy (Equation 6), would have been used. The reason for this is because:

\[
N_e e^{-0.0349 \beta \tan \phi} \frac{1}{N_q - 1} \approx \cos \beta \left( e^{-2\beta \tan \phi} - \frac{2\beta}{2 + \pi} e^{-\pi \tan \phi} \right)
\]  
\( (12) \)

The results of this equation, the German design norm and the Bishop’s slip circle method have been plotted in Figure 5, together with the results from the Finite Element calculations. This figure shows that the analytical approximation functions reasonably with a reduced logarithmic spiral-wedge, which is according to the numerical calculations not the case. Also plots of the incremental displacements of the FEM calculations, indicating this failure mechanism, show that this approach is not correct for purely frictional soil (Figure 6).

4.2 Soil-weight slope factor \( \lambda_c \)

A mistake which can be found in the publication of Meyerhof (1957), but also in many recent publications, is the assumption that the failure mechanism in a purely frictional soil \( (N_f) \), is a Prandtl-wedge

\[
\lambda_f = 1 - \left( \frac{\beta}{\phi} \right)^{\frac{1}{2}} \quad \beta \leq \phi
\]  
\( (13) \)

The results of this equation, the German design norm and the Bishop’s slip circle method have been plotted in Figure 7, together with the results from the Finite Element calculations. This figure shows that the analytical approximation functions reasonably

Figure 5. Cohesion slope factor (Analytical solution, German norm and Bishop versus FEM).

Figure 6. Failure mechanism: Prandtl-wedge versus FEM (Incremental displacement plots).

Figure 7. Soil-weight slope factor (Analytical solution, German norm and Bishop versus FEM).
well. The German design norm does not fit. The Bishop calculations do not fit at all. An important reason for this is the fact that the slip circle in the Bishop calculations has been forced not to cross the foundation plate, while the FEM calculations show that the soil slides somewhere below the plate (see especially Figure 6 for $\beta = 20^\circ$), which causes the plate to tumble over. This tumbling failure mechanism however, is not part of the Bishop calculation method.

\[ \lambda_q = e^{-2\beta \tan \phi} \quad \beta \leq \phi, \quad (14) \]

which is the same equation for the surcharge slope factor as the one proposed by Ip (2005). The problems with this solution are:

- it neglects the extension (dashed lines) due to the depth, and
- the surcharge slope factor is not “0” for $\beta = \phi$, which is the same problem as for the slope factors of Meyerhof in Figure 3.

In order to see the influence of having a shallow solid footing, additional finite element calculations have been made for a relative depth of $D/B = 1$. This relative depth creates an additional bearing capacity mostly due to the surcharge of $q = \gamma D$, but also due to a larger slip surface, of which the influence is difficult to quantify.

Plots of the incremental displacements of the FEM calculations, indicating the failure mechanism, show that the failure mechanism of a shallow foundation in a frictional soil with self-weight, is an extended Prandtl-wedge with a reduced logarithmic spiral-wedge, see Figure 8.

Since Figure 8 shows that in general the failure mechanism looks like an extended Prandtl-wedge with a reduced logarithmic spiral-wedge, it seems possible to derive an analytical solution for the surcharge slope factor, which is purely based on the reduced logarithmic spiral-wedge. This would give the following surcharge slope factor:

\[ \lambda_q = 1 - \left( \frac{\beta}{\phi} \right)^{3/2} \quad \beta \leq \phi, \quad (15) \]

The results of this equation, the German design norm and the Bishop’s slip circle method have been plotted in Figure 9, together with the results from the Finite Element calculations.

This figure shows that the Bishop calculations fit better for steeper slopes this time, but still not for gentle slopes. The German design norm and especially the analytical approximation are functioning reasonably well.
5 CONCLUSIONS

A large number of finite element calculations of strip footings next to a slope have been made in order to check the failure mechanisms, and to check the bearing capacities, first for the currently used equations for the slope bearing capacity factors, proposed by the German Annex of the Geotechnical Eurocode, and second for the bearing capacity calculated with the Bishop slip circle stability calculation method. These calculations prove that both the German slope factors and the Bishop calculations are often inaccurate.

Therefore new factors have been proposed, which are for both the soil-weight and the surcharge slope bearing capacity, based on the numerical calculations, and for the cohesion slope bearing capacity, also on an analytical solution.

6 REFERENCES


Meyerhof, G. G. (1951) The ultimate bearing capacity of foundations, Géotechnique, 2, pp. 301-332


