Travaux mathématiques, Volume 20 (2012), 15–27, © Université du Luxembourg

Polynomial separating overgroups

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Abstract

We first prove that the moment map for a unitary representation of a Lie group G, defined by N. J. Wildberger is a geometric moment map, coming from a strongly Hamiltonian action of G on a real Fréchet symplectic manifold. Then we define a Fréchet Lie group $\widetilde{G}$, containing G and extensions of each irreducible unitary G-representation into an affine $\widetilde{G}$-action, whose moment map characterizes the unitary representation.

Then we look for construction of overgroups G^+, *i. e.* Lie groups containing G and extensions of the generic coadjoint orbits, resp. unitary representations of G to corresponding objects for G^+, by using a quadratic function. We consider here the cases G nilpotent, for which this is possible if $\dim G \leq 7$, and the case $G = SL(n, \mathbb{R})$, where such a construction exists only if $n = 2$.

1 Introduction

This lecture is the presentation of a common work, with Mohamed Selmi and Amel Zergane from the Sousse University (Tunisia).

N. J. Wildberger introduced the moment map for a unitary representation $(\mathcal{H}, \pi)$ of a Lie group G (see [17, 18]). He proved that, for compact Lie group and irreducible π, the range of this map characterizes the representation π (see also [14]). Then he studied the nilpotent case for which, if π is irreducible, the range is the closed convex hull of the coadjoint orbit $\mathcal{O}$, associated to π. But they are example of distinct orbits having same convex hull.

In a series of papers ([4, 9, 3, 1]), L. Abdelmoula, A. Baklouti, J. Ludwig, M. Selmi, and myself try to characterize the representation π through an extension of the moment map to the whole universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G.

In the last period, with Mohamed Selmi and Amel Zergane, (see [5, 6, 7, 19]) we prefer to extend the moment map to a larger group G^+, containing G. We call such a group an overgroup for G. We expect to keep in this extension a geometric interpretation of the moment map. But we hope also to define these extensions by using mapping as regular as possible, that means polynomial mapping and, if it is possible, quadratic mappings.

Here, we first prove that the Wildberger moment map is a geometric moment map, coming from a strongly Hamiltonian action of G on a Fréchet symplectic manifold, namely $(\mathbb{P}\mathcal{H}^\infty)_\mathbb{R}$. This needs good definitions for Fréchet differential calculus and Fréchet manifolds. Then we can define Fréchet Lie groups, and build what we call the universal overgroup for G, which is a Fréchet Lie group $\widetilde{G}$, containing G and extensions of the irreducible unitary G-representations π into affine actions $\widetilde{\pi}$, such that the moment map of $\widetilde{\pi}$ characterizes π.

In a second part of the lecture, we look for the construction of Lie overgroups G^+, semi direct product $G^+ = G \ltimes V$, where V is a finite dimensional module and extensions of the generic coadjoint orbits, resp. generic unitary representations of G to corresponding objects for G^+, by using a polynomial or even a quadratic function $\varphi : \mathfrak{g}^\star \longrightarrow (\mathfrak{g}^+)^\star$. We want to present here the main results in this direction, thus we will restrict ourselves to the two cases G nilpotent, connected and simply connected, and $G = SL(n,\mathbb{R})$.

In the first case, we present different constructions for such quadratic overgroups, holding for different classes of nilpotent groups G. Then we prove that if G is nilpotent and $\dim G \leq 7$, it is possible to build such a group G^+, and extensions, which characterize the generic coadjoint orbits $\mathcal{O}$ in $\mathfrak{g}^\star$ by the convex hull of the corresponding coadjoint orbit $\mathcal{O}^+ = \varphi(\mathcal{O})$ in $(\mathfrak{g}^+)^\star$.

In the case $G = SL(n,\mathbb{R})$, since it is possible to describe explicitely finite dimensional modules, we prove this program is possible if and only if $n = 2$. Moreover, there is such a construction with a polynomial mapping φ, with $\deg(\varphi) = n$, thus there is a cubic overgroup for $n \leq 3$, but not for $n = 4$.

2 Moment for a representation

2.1 Linear and projective actions

Let G be a Lie group and $(\mathcal{H},\pi)$ a unitary representation of G. Let us first suppose that $\mathcal{H}$ is a finite dimensional vector space. Then the underlying real space $\mathcal{H}_\mathbb{R}$ is a symplectic vector space for the form:

$$\omega^\mathcal{H}(w_1,w_2) = \mathfrak{Im}(w_1|w_2).$$

Of course, this form $\omega^\mathcal{H}$ is invariant under the linear action $(g,v) \mapsto \pi(g)v$.

Similarly, the real manifold $(\mathbb{P}\mathcal{H})_\mathbb{R}$ (*i. e.* the complex projective space considered as a real manifold) is a symplectic manifold:

First it is a smooth manifold, equipped with the following local charts φ_v. Let v be any non vanishing vector in $\mathcal{H}$, put

$$U_v = \{[w] \in \mathbb{P}\mathcal{H}, \text{ such that } (v|w) \neq 0\},$$

and

$$\varphi_v : U_v \longrightarrow (v^\perp)_\mathbb{R}, \qquad \varphi_v : [w] \mapsto \|v\|^2 \frac{w}{(v|w)} - v.$$

These maps are smooth, bijective and $\varphi_v^{-1}(u) = [v + u]$. The family of all these charts form an atlas for the manifold $(\mathbb{P}\mathcal{H})_\mathbb{R}$, moreover a direct computation shows that the formulas:

$$\omega_{[v]}^{\mathbb{P}\mathcal{H}}(W_1, W_2) = \mathfrak{Im}\frac{(d\varphi_v(W_1)|d\varphi_v(W_2))}{\|v\|^2}$$

defines a smooth 2-form on the manifold $(\mathbb{P}\mathcal{H})_\mathbb{R}$, and this form is closed and everywhere non degenerated.

Now, G acts on $(\mathbb{P}\mathcal{H})_\mathbb{R}$ by $(g, [v]) \mapsto [\pi(g)v]$, this action is smooth, and $\omega^{\mathbb{P}\mathcal{H}}$ is invariant.

These two actions, the linear and the projective one, are strongly Hamiltonian: for each X in the Lie algebra $\mathfrak{g}$ of G, there is a smooth function J_X on $\mathcal{H}_\mathbb{R}$ (resp. on $(\mathbb{P}\mathcal{H})_\mathbb{R}$) such that, for any smooth function f,

$$\{J_X, f\} = \frac{d}{dt}\Big|_{t=0} f(\pi(\exp -tX)\cdot),$$

moreover these functions J_X can be chosen in such a manner that, for any X and Y,

$$\{J_X, J_Y\} = J_{[X,Y]}.$$

The corresponding moment maps ψ_π and Ψ_π are the following mappings, $\psi_\pi : \mathcal{H}_\mathbb{R} \longrightarrow \mathfrak{g}^\star$ (resp. $\Psi_\pi : (\mathbb{P}\mathcal{H})_\mathbb{R} \longrightarrow \mathfrak{g}^\star$):

$$J_X(v) = \langle \psi_\pi(v), X\rangle = \frac{1}{2i}(\pi'(X)v|v),$$

$$\Big(\text{resp. } J_X([v]) = \langle \Psi_\pi([v]), X\rangle = \frac{1}{2i}\frac{(\pi'(X)v|v)}{\|v\|^2}\Big).$$

2.2 Moment set for π

In any case, *i. e.* $\dim \mathcal{H}$ finite or infinite, N. J. Wildberger ([17, 18]) defined

Definition 2.1. The moment set $\mathcal{I}_\pi$ for π is the closure in $\mathfrak{g}^\star$ of the set:

$$\{\Psi_\pi([v]), \quad v \in \mathcal{H}^\infty \setminus \{0\}\}.$$

Remark this map is something like a dequantization procedure. Indeed, suppose for instance G is an exponential Lie group, then there is an one-to-one, onto map from the space $\mathfrak{g}^\star/G$ of coadjoint orbit in $\mathfrak{g}^\star$ and the unitary dual $\widehat{G}$ of the group G ([10]). This map can be considered as a quantization of each coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^\star$ ([13]).

Here, we consider a map in the “opposite” direction, from $\widehat{G}$ to the family $\mathcal{C}$ of closed subsets in $\mathfrak{g}^\star$. Unfortunately, the map $\pi \mapsto \mathcal{I}_\pi$ is not directly the inverse of the map $\mathcal{O} \mapsto \pi$.

Anyway, let us recall some known facts about this map:

- If G is compact, the map $\pi \mapsto \mathcal{I}_\pi$ is one-to-one: the moment set $\mathcal{I}_\pi$ characterizes the representation π ([14, 17]),
- If G is solvable, $\mathcal{I}_\pi$ is convex, generally, $\pi \mapsto \mathcal{I}_\pi$ is not injective ([4]),
- Very generally, for irreducible π, $\mathcal{I}_\pi = \overline{\mathrm{Conv}}(\mathcal{O})$, the closed convex hull of a coadjoint orbit $\mathcal{O}$, but, even if G is nilpotent, $\mathcal{O} \mapsto \mathrm{Conv}(\mathcal{O})$ is not one-to-one ([18]),
- If we extend the map Ψ_π to the universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$ of $\mathfrak{g}$, these extensions define an injective mapping from $\widehat{G}$ to $(\mathfrak{A}(\mathfrak{g}))^\star$ ([1]).

2.3 Schedule of the lecture

Today, we look for two different goals.

1. We want to view the Wildberger maps ψ_π and Ψ_π as true geometric moment maps. This needs use of infinite dimensional manifolds, on Fréchet spaces. Therefore, we look for Fréchet differential calculus and Fréchet manifolds as defined by R. S. Hamilton in [12].

   Then, with this notion of Fréchet manifolds, we can build a Fréchet Lie-group, semi-direct product $\widetilde{G} = G \rtimes V$ of G with a real Fréchet vector space, and extend each unitary irreducible representation π of G to an affine action $\widetilde{\pi} = \Phi(\pi)$ of $\widetilde{G}$, which is Hamiltonian and such that the moment $\mathcal{I}_{\widetilde{G}}$ for this action does characterize π: we say that $(\widetilde{G}, \Phi)$ is a $\widehat{G}$-separating overgroup of G.

   This construction is very general, but uses a very "large" infinite dimensional Fréchet Lie group $\widetilde{G}$, we call this group, the universal overgroup for G.

2. Of course, we prefer to work with (finite dimensional) Lie groups, so we will restrict ourselves to Lie overgroups, semi-direct products $G^+ = G \rtimes V$ and look to the existence of a polynomial mapping $\phi : \mathfrak{g}^\star \longrightarrow V^\star$ such that, if $\varphi(\ell) = (\ell, \phi(\ell))$, we get, for any generic ℓ in $\mathfrak{g}^\star$:

   $$\varphi(G\cdot\ell) = G^+\cdot\varphi(\ell), \quad \text{and} \quad \mathrm{Conv}(\varphi(G\cdot\ell)) = \mathrm{Conv}(\varphi(G\cdot\ell')) \Longrightarrow G\cdot\ell = G\cdot\ell'.$$

   If such a ϕ exists, we say that (G^+, φ) is a polynomial $\mathfrak{g}^\star/G$-separating overgroup for G. If we get separation for a large subset $\mathfrak{g}^\star_{gen}/G$ of $\mathfrak{g}^\star/G$, we say that (G^+, φ) is a polynomial $\mathfrak{g}^\star_{gen}/G$-separating overgroup for G.

   If it is possible to choose ϕ, with $\deg(\phi) \leq 2$, we just say that G admits a quadratic overgroup, or (G^+, φ) is a quadratic overgroup for G.

Existence of such overgroups seems to be a very restrictive condition for the structure of the group G, but we shall present here the nilpotent case, for which the quadratic condition is in fact not too restrictive and the $SL(n, \mathbb{R})$ case, for which the quadratic condition is very strict.

3 Fréchet differential geometry

3.1 Fréchet differential calculus

Let U be an open subset in a real Fréchet vector space V and W another real Fréchet space. Following R. S. Hamilton ([12]), we say that a continuous function $f : U \longrightarrow W$ is derivable in the direction $h \in V$ if the following limit exists:

$$\lim_{t\to 0} \frac{1}{t} \left(f(u+th) - f(u) \right) = Df(u)(h) = \langle \nabla f|_u, h \rangle.$$

We shall say that f is C^1 if $(u,h) \mapsto Df(u)(h)$ is defined and continuous from $U \times V$ into W. Similarly, f is C^2 if, for any h_1, $u \mapsto Df(u)(h_1)$ is continuous, derivable in any direction h_2, and $(u,h_1,h_2) \mapsto (D^2f)(u)(h_1,h_2)$ is continuous from $U \times V \times V$ into W, and so one ...

The Schwarz lemma, the chain rule hold, but there is no local inverse function theorem as the following example (see [12]) shows:

Let V be the space $C^\infty(\mathbb{R})_1$ of smooth real functions u on $\mathbb{R}$, periodic, with period 1 (with its natural Fréchet topology). Let $F : \mathbb{R} \times V \longrightarrow V$ be the function defined by:

$$(t,u) \mapsto F(t,u)(x) = \int_0^1 u(x+ts)\ ds.$$

We can verify that F is a C^∞ map, we have $F(0,0) = 0$ and $F(0,u) = u$. Then the partial derivative $\frac{DF}{Du}|_{u=0}$ is the invertible map id_V. But fix n, and put, for any k, $u_k(x) = \sin(2kn\pi x)$. We get for any k

$$F(\frac{1}{n}, u_k) = 0,$$

and $u \mapsto F(\frac{1}{n}, u)$ is not one-to-one. If n varies, there is no open subset containing 0 such that the equation $F(t,u) = 0$ has an unique solution.

3.2 Fréchet symplectic manifolds

A Fréchet manifold is a Hausdorff space with local charts $\varphi_i : U_i \longrightarrow E$ (where E is a Fréchet space), such that

Each φ_i is a homeomorphism from U_i onto an open subset in E, and,

For any i and j, $\varphi_j \circ \varphi_i^{-1}$ is a homeomorphism and a C^∞ map between two open subsets in E.

We define C^∞ functions, vector fields and forms on a Fréchet manifold as for a finite dimensional manifold: the smoothness of these quantities can be tested in each local chart.

Let us now come to the case $V = (\mathcal{H}^\infty)_\mathbb{R}$, the real space of C^∞ vectors for a unitary representation $(\mathcal{H}, \pi)$ of the Lie group G. Put

$$\omega^V(w_1, w_2) = \mathfrak{Im}(w_1|w_2).$$

It is a non degenerated, bilinear, antisymmetric form, constant, thus C^∞. Consider now the flat-map:

$$\flat : V \longrightarrow V^\star, \qquad \langle u^\flat, v\rangle = \mathfrak{Im}(u|v).$$

Denote $V^\flat$ the range of this map. On $V^\flat$, we put the topology coming from V through the bijective map $\flat$. If $\dim \mathcal{H}$ is infinite, then $V^\flat \subsetneq V^\star$.

Take for instance for G the Heisenberg group, and $(\mathcal{H}, \pi)$ the usual unitary irreducible representation of G, onto $\mathcal{H} = L^2(\mathbb{R})$. Then V is the Schwartz space $\mathcal{S}(\mathbb{R})$, and

$$V^\flat = \mathcal{S}(\mathbb{R}) \neq V^\star = \mathcal{S}'(\mathbb{R}).$$

Therefore, we define Hamiltonian functions as:

Definition 3.1. A C^∞ function on V is a Hamiltonian function, if ∇f is C^∞ from V into $V^\flat$.

Suppose $V = \mathcal{S}(\mathbb{R})$, as above and f is the linear function $f : u \mapsto u(0)$, f being linear and continuous is C^∞, but it is not a Hamiltonian function, since $\nabla f = \delta_0$ is the Dirac distribution.

Denote now $\mathbb{P}V$ the set $(\mathbb{P}\mathcal{H}^\infty)_\mathbb{R}$. With the local charts (U_v, φ_v) defined as above, it is a C^∞ manifold, we put

$$\omega^{\mathbb{P}V}_{[v]}(W_1, W_2) = \frac{\mathfrak{Im}(d\varphi_v(W_1)|d\varphi_v(W_2))}{\|v\|^2}.$$

This formula defines a well defined, C^∞ 2-form on $\mathbb{P}V$. This 2-form is non degenerated at any point and it is closed:

We just say that $\mathbb{P}V$ is a symplectic Fréchet manifold.

We say that a C^∞ function on $\mathbb{P}V$ is Hamiltonian if it is Hamiltonian in each local chart.

3.3 Linear and projective actions

Consider now the linear and projective actions of G on the symplectic Fréchet manifolds V and $\mathbb{P}V$.

Proposition 3.2. *([8])*

These two actions preserves the corresponding forms, moreover they are strongly Hamiltonian with the Wildberger Hamiltonian functions:

$$v \mapsto \langle \psi_\pi(v), X\rangle, \qquad [v] \mapsto \langle \Psi_\pi([v]), X\rangle.$$

Indeed, a direct computation shows these functions are well defined, C^∞ and Hamiltonian functions, they generate vector fields $\flat^{-1}(\nabla\psi_\pi)$, respectively $\flat^{-1}(\nabla\Psi_\pi)$ which are the infinitesimal generators for the linear, respectively projective actions.

3.4 Fréchet Lie group, universal overgroup

Let us come to the notion of Fréchet Lie group. It is a smooth manifold and a group, such that the maps $(g_1, g_2) \mapsto g_1 g_2$ and $g \mapsto g^{-1}$ are C^∞.

For instance, if $(\mathcal{H}, \pi)$ is a unitary representation of G, if $V = (\mathcal{H}^\infty)_\mathbb{R}$, thus $G \rtimes V$ is a Fréchet Lie group, with $\mathfrak{g} \rtimes V$ as Lie algebra. The product and the Lie bracket are:

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + \pi(g_1)v_2),$$

and

$$[(X_1, v_1), (X_2, v_2)] = ([X_1, X_2], \pi'(X_1)v_2 - \pi'(X_2)v_1)\,.$$

Consider now the space $\mathcal{H} = \oplus_{\pi\in\widehat{G}}\mathcal{H}_\pi$ and $V = (\mathcal{H}^\infty)_\mathbb{R}$, denote $\widetilde{G}$ the group $G \rtimes V$.

For any π_0 in $\widehat{G}$, we note p_{π_0} the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\pi_0}$. We finally extend the linear action π_0 of G into an affine action of $\widetilde{G}$ on $V_{\pi_0} = (\mathcal{H}^\infty_{\pi_0})_\mathbb{R}$ by putting:

$$\widetilde{\pi}_0(g, u)v = \pi_0(g)v + p_{\pi_0}(u) \qquad\qquad (u \in V, \quad v \in V_{\pi_0}).$$

This affine action is Hamiltonian, but not strongly Hamiltonian, its moment map (vanishing at the identity) is

$$\psi_{\widetilde{\pi}_0}(v)(X, u) = \frac{1}{2}\mathfrak{Im}(\pi_0'(X)v|v) + \mathfrak{Im}(p_{\pi_0}(u)|v).$$

Denote $\mathcal{I}_{\widetilde{\pi}_0}$ the corresponding moment set and q the canonical projection from $\widetilde{\mathfrak{g}}^\star$ onto $\mathfrak{g}^\star$. Then

$$q(\mathcal{I}_{\widetilde{\pi}_0}) = \mathcal{C}(\mathcal{I}_{\pi_0}) = \text{ Cone with base } \mathcal{I}_{\pi_0}.$$

Finally, $\mathcal{I}_{\widetilde{\pi}_0}$ characterizes the representation π_0:

$$\mathcal{I}_{\widetilde{\pi}_1} = \mathcal{I}_{\widetilde{\pi}_2} \quad\Longrightarrow\quad \pi_1 = \pi_2.$$

So we get

Theorem 3.3. *([8])*

The Fréchet Lie group $\widetilde{G}$ and the extensions $\Phi : \pi \mapsto \widetilde{\pi}$ of the linear actions π to affine actions $\widetilde{\pi}$ define a universal overgroup for G, i. e. a Fréchet Lie group, which is $\widehat{G}$-separating.

4 Polynomial Lie overgroups

Recall we are looking for Lie groups of the form $G^+ = G \rtimes V$ where $\dim V$ is a finite dimensional vector space, and polynomial mapping $\phi : \mathfrak{g}^\star \longrightarrow V^\star$ (thus the polynomial map $\varphi(\ell) = (\ell, \phi(\ell))$ from $\mathfrak{g}^\star$ into $(\mathfrak{g}^+)^\star$), such that, for any generic ℓ_i in $\mathfrak{g}^\star$,

$$\varphi(G \cdot \ell_1) = G^+ \cdot \varphi(\ell_1),$$

and

$$\mathrm{Conv}(G^+ \cdot \varphi(\ell_1)) = \mathrm{Conv}(G^+ \cdot \varphi(\ell_2)) \quad \Longrightarrow \quad G \cdot \ell_1 = G \cdot \ell_2,$$

we say that (G^+, φ) is a polynomial $\mathfrak{g}^\star_{gen}/G$-separating overgroup for G.

Let us consider the two 'extremal' cases G nilpotent and $G = SL(n, \mathbb{R})$.

4.1 The nilpotent case

Suppose now G is nilpotent, connected and simply connected. Let us first define generic points ℓ in $\mathfrak{g}^\star$. Fix a Jordan-Hölder basis $(e_1, \dots, e_n)$ in $\mathfrak{g}^\star$, with respect to the coadjoint action, for any subset K of $\{1, \dots, n\}$, denote V_K the vector space generated by the e_k, for k in K.

For each ℓ in $\mathfrak{g}^\star$, there is a subset $J \subset \{1, \dots, n\}$, called the set of jump indices for the orbit $G \cdot \ell$, they are the direction where the orbit growths. More precisely (see [16]), if $J' = \{1, \dots, n\} \setminus J$,

1. The restriction, to the orbit $G \cdot \ell$, of the projection onto V_J, parallel to $V_{J'}$ is a global diffeomorphism: $G \cdot \ell \simeq V_J$,
2. The intersection $G \cdot \ell \cap V_{J'}$ is a singleton: $G \cdot \ell \cap V_{J'} = \{\lambda(\ell)\}$.

Consider now the subset $\mathfrak{g}^\star_{gen}$ of the points ℓ in $\mathfrak{g}^\star$, such that J is minimal for the lexicographic ordering. This set is an invariant, dense, Zariski open subset in $\mathfrak{g}^\star$. On this set, the function λ is rational, and take the form:

$$\lambda(\ell) = \sum_{k \notin J} \frac{P_k(\ell)}{Q_k(\ell)} e_k.$$

The functions P_k are in fact invariant polynomial functions and they generate the field $R(\mathfrak{g})$ of rational invariant functions on $\mathfrak{g}^\star$:

$$R(\mathfrak{g}) = \mathbb{R}(P_k).$$

Thanks to this well known construction (see for instance [16]), we re-find here a result of Baklouti, Ludwig and Selmi in [9]:

Proposition 4.1. *Put $\phi(\ell) = \sum_{k \notin J} P_k(\ell) e_k$, then $(G \times V_{J'}, \varphi)$ (the action of G on $V_{J'}$ is trivial) is a polynomial $\mathfrak{g}^\star_{gen}/G$-separating overgroup for G.*

Since $\widehat{G}$ is homeomorphic to the space of orbits in $\mathfrak{g}^\star$, we can write this at the level of representations:

Define $\Phi : (\widehat{G})_{gen} \longrightarrow \widehat{G^+}$ as the map associating to each generic, irreducible unitary representation π of G (π is associated to a generic orbit) its extension to G^+ defined by $\pi'(e_k) = iP_k(\ell)$. Then (G^+, Φ) is a $(\widehat{G})_{gen}$-separating overgroup for G.

Especially, if the maximum of the degree of the P_k is at most 2, G admits a quadratic overgroup.

On the other hand, there is a different way to build quadratic overgroup for G. Let us present here this method:

1. We say that G is *special* if there is an abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$, with codim $\mathfrak{a} = 1/2\ \#J$.

2. If G is special, then its generic coadjoint orbits are fibre bundles over the G-orbit in $\mathfrak{a}^\star$, with $\mathfrak{a}^\perp$ as fiber. Thus, we can rebuild the orbit, starting with the G-orbit in $\mathfrak{a}^\star$.

3. Now the map $\theta : \ell|_\mathfrak{a} \mapsto (\ell|_\mathfrak{a})^2$ from $\mathfrak{a}^\star$ into $S^2(\mathfrak{a})$ is strictly convex, that means, if p is the natural projection from $\mathfrak{a}^\star \oplus S^2(\mathfrak{a})$ onto $\mathfrak{a}^\star$ and $\vartheta(f) = (f, \theta(f))$, for any subset A in $\mathfrak{a}^\star$ (see [6]),
$$p\left(\overline{\text{Conv}}\ (\vartheta(A)) \cap \vartheta(\mathfrak{a}^\star)\right) = \overline{A}.$$

4. Consider $G^+ = G \rtimes S^2(\mathfrak{a}^\star)$, define $\varphi : \mathfrak{g}^\star \longrightarrow (\mathfrak{g}^+)^\star$ by $\varphi(\ell) = (\ell, \theta(\ell|_\mathfrak{a}))$, a direct computation shows that, for any generic ℓ, $\varphi(G \cdot \ell) = G^+ \cdot \varphi(\ell)$.

Proposition 4.2. *([6, 7])*
A special nilpotent Lie group G admits $(G^+ = G \rtimes S^2(\mathfrak{a}), \varphi)$ as quadratic overgroup. More precisely, this overgroup is $\mathfrak{g}^\star_{gen}/G$-separating.

Moreover, for any generic representation $\pi \in \widehat{G}_{gen}$, associated to the coadjoint orbit $\mathcal{O}$, if $\Phi(\pi)$ is the representation of G^+ associated to the orbit $\varphi(\mathcal{O})$, then $(G^+ = G \rtimes S^2(\mathfrak{a}), \Phi)$ is $\widehat{G}_{gen}$-separating.

Considering the classification of small dimensional nilpotent Lie algebras (see [11, 15]), we see that all the nilpotent Lie groups, with $\dim G \leq 6$, except one called $G_{6,20}$ either are special or verify $\max\deg P_k \leq 2$, thus admit a quadratic overgroup.

Finally, we generalize the spacial case, by considering the case of two ideals in $\mathfrak{g}$: $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{g}$, with $\mathfrak{b}$ special, $\mathfrak{a}$ being the abelian ideal given in the '$\mathfrak{b}$ special' definition. We moreover suppose that, for generic ℓ, $\mathfrak{b} + \mathfrak{g}(\ell) = \mathfrak{g}$. With these hypothesis, the G orbits are diffeomorphic to the B-orbits in $\mathfrak{b}^\star$, moreover, if $\lambda_\mathfrak{g}$ and $\lambda_\mathfrak{b}$ are the functions λ defined above, but for the Lie algebras $\mathfrak{g}$ and $\mathfrak{b}$, then any convex combination of points in $G \cdot \lambda_\mathfrak{g}(\ell)$, which are such that $\ell|_\mathfrak{b} = \lambda_\mathfrak{b}(\ell|_\mathfrak{b})$ takes exactly the value $\lambda_\mathfrak{g}(\ell)$. Therefore,

Proposition 4.3. *([2])*

For a nilpotent Lie group G satisfying the above conditions, $(G^+ = G \rtimes S^2(\mathfrak{a}), \varphi)$, where $\varphi(\ell) = (\ell, (\ell|_\mathfrak{a})^2)$, is a quadratic overgroup. More precisely, this overgroup is a $\mathfrak{g}^\star_{gen}/G$-separating group.

Remark that we cannot separate the closure of the convex hull of the orbits, thus we cannot separate the irreducible unitary representation by their moment sets.

However, it is possible to verify, case by case, that any nilpotent Lie group G, with $\dim G \leq 7$ satisfies the hypothesis of one of our proposition, thus admits a quadratic overgroup.

Last remark: there is a 12-dimensional example of a Lie group, whose invariants are not generated by quadratic functions, which is not special, and not in the last class of groups. This example admits a quadratic overgroup.

Indeed, it is probably impossible to prove that a given nilpotent Lie group has no quadratic overgroup, just because there is no classification of finite dimensional modules for a nilpotent Lie algebra. For this reason, we consider now a totally different setting, the case $G = SL(n, \mathbb{R})$.

4.2 The $SL(n, \mathbb{R})$ case

Recall well known facts about the $SL(n, \mathbb{R})$-coadjoint orbits:

1. Thanks to the Killing form, we identify adjoint and coadjoint actions.
2. The set of generic ℓ is the set of $n \times n$ matrices, with n distinct eigenvalues. This set is dense and open in $\mathfrak{sl}(n, \mathbb{R})$.
3. If $n \geq 3$, then any generic orbit satisfies $\overline{\text{Conv}}(\mathcal{O}) = \mathfrak{sl}(n, \mathbb{R})$.
4. The invariant polynomial functions on $\mathfrak{sl}(n, \mathbb{R})$ are polynomials in the functions t_k, for $k = 2, \ldots, n$:
$$t_k(\ell) = Tr(\ell^k).$$
5. These functions separate almost the generic orbits, the only case where there is no separation, is the case n even, and orbits of matrices ℓ having only non real eigenvalues. In this case, there are exactly 2 orbits $G \cdot \ell_1$, and $G \cdot \ell_2$ on which the invariant functions take the same values: ℓ_1 and ℓ_2 have the same spectrum, but are conjugated through a matrix with determinant -1.

Thus we can say:

Proposition 4.4. *$SL(n, \mathbb{R})$ admits an almost $\mathfrak{sl}(n, \mathbb{R})^\star_{gen}$-separating polynomial overgroup, with degree n, namely $(G^+ = SL(n, \mathbb{R}) \times \mathbb{R}^{n-1}, \varphi)$, with*

$$\varphi(\ell) = (\ell, t_2(\ell), \ldots, t_n(\ell)).$$

Now, suppose there is a quadratic overgroup for $SL(n,\mathbb{R})$, then, using semi simplicity of finite dimensional real $\mathfrak{sl}(n,\mathbb{R})$-modules, A. Zergane can prove there is such a quadratic overgroup of the form

$$G^+ = SL(n,\mathbb{R}) \rtimes (\mathfrak{sl}(n,\mathbb{R}) + S^2(\mathfrak{sl}(n,\mathbb{R}))), \qquad \phi(\ell) = b_1(\ell) + b_2(\ell,\ell),$$

where b_1 and b_2 are intertwining operators respectively for the modules $\mathfrak{sl}(n,\mathbb{R})^\star$ and $S^2(\mathfrak{sl}(n,\mathbb{R}))^\star$.

Let us now compute all these intertwining operators: if $n > 3$, there are the trace operators defined as:

$$\langle P_0(\ell), X\rangle = Tr(\ell X),$$

$$\langle P_1(\ell.\ell'), X.X'\rangle = Tr(\ell X\ell' X') + Tr(\ell X'\ell' X),$$

$$\langle P_2(\ell.\ell'), X.X'\rangle = Tr(\ell X)Tr(\ell' X') + Tr(\ell X')Tr(\ell' X),$$

$$\langle P_3(\ell.\ell'), X.X'\rangle = Tr(\ell\ell' XX') + Tr(\ell\ell' X'X) + Tr(\ell'\ell XX') + Tr(\ell'\ell X'X),$$

$$\langle P_4(\ell.\ell'), X.X'\rangle = Tr(\ell\ell')Tr(XX').$$

If $n = 3$, P_3 is a linear combination of P_1, P_2, P_4.

Finally, looking to the condition $G^+\cdot\varphi(\ell) = \varphi(SL(n,\mathbb{R})\cdot\ell)$, A. Zergane proves the only possibilities are in fact non separating:

Proposition 4.5. *([19])*
If $n \geq 3$, $SL(n,\mathbb{R})$ does not admit any quadratic overgroup.

If $n = 2$, we found a quadratic overgroup. Similarly, $SL(3,\mathbb{R})$ admits a cubic overgroup and we can prove, with the same method, that $SL(4,\mathbb{R})$ does not admit any cubic overgroup.

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