Travaux mathématiques, Volume 20 (2012), 29–82,

Atiyah Classes and Equivariant Connections on Homogeneous Spaces

by Martin Bordemann

Abstract

We shall give a 'pedagogical' review of G-invariant connections on not necessarily reductive homogeneous spaces, its obstructions -due to Nguyen-van Hai, 1965- leading to the Atiyah classes recently dealt with in the literature, and applications to multi-differential operators on homogeneous spaces.

Introduction

In recent years, the interest in not necessarily reductive homogeneous spaces seems to have increased: among other things, the reason is a new attack on an old problem by Michel Duflo where algebras of invariant differential operators are compared to the algebras of their symbols, see e.g. [14], [10], [11]. Moreover, in the preprint [7], Calaque, Căldăraru, and Tu observed that for any Lie algebra inclusion $\mathfrak{h} \subset \mathfrak{g}$ the $\mathfrak{h}$-module $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$ is isomorphic (as a filtered module) to the $\mathfrak{h}$-module $\mathsf{S}(\mathfrak{g}/\mathfrak{h})$ iff a certain cohomology class of rank 1 in the Chevalley-Eilenberg cohomology of $\mathfrak{h}$ vanishes which they called the *Atiyah class* of the Lie algebra inclusion. For a particular case, where $\mathfrak{g}$ is the Lie algebra of all vector fields on a coisotropic submanifold C of a symplectic manifold, and $\mathfrak{h}$ is the subalgebra of all vector fields on C along the canonical foliation the author has observed that this class –which had been defined by P.Molino in 1971, see [29], [30]– was related to obstructions of the representability of a star-product on the ambient symplectic manifold on the space of smooth functions on C, see the preprint [5] and the proceeding [6]. Similar results to [7] have been extended to Lie algebroids, see [12] and [8].

The aim of this proceeding is to relate the above observations to a classical subject in the theory of homogeneous spaces, namely the question of whether a homogeneous space admits invariant connections in a G-equivariant principal bundle over the space. This had already been done in a work by H.-C. Wang, [36] in 1958, but where the cohomological nature of the existence of these connections had not explicitly been mentioned: in those days, the main focus seemed to have been the study of compact or more generally reductive homogeneous spaces for which invariant connections always exist. In 1965 Nguyen-van Hai [31] formulated the cohomological obstruction now called the Atiyah-class for the case of a linear connection which has been rediscovered in [7]. With the coadjoint orbits, a lot of examples of nonreductive homogeneous spaces have been studied. For instance, the work of Pikulin and Tevelev [33] deals with the question of invariant symplectic connections on nilpotent coadjoint orbits of reductive groups.

The main idea of the Atiyah class is very simple: let

$$\{0\} \to (\mathsf{A}, d_A) \to (\mathsf{B}, d_B) \to (\mathsf{C}, d_C) \to \{0\}$$

be an exact sequence of nonnegatively graded cocomplexes (i.e. the degree of the differentials is +1). According to classical homological algebra there is the associated long exact cohomology sequence

$$\{0\} \to H^0(\mathsf{A}) \to H^0(\mathsf{B}) \to H^0(\mathsf{C}) \stackrel{\text{connecting hom}}{\longrightarrow} H^1(\mathsf{A}) \to H^1(\mathsf{B}) \to \cdots$$

It is immediate that a class $[\gamma]$ in $H^0(\mathsf{C})$ lifts to a class in $H^0(\mathsf{B})$ iff its image $c_{[\gamma]}$ under the connecting homomorphism in $H^1(\mathsf{A})$, which we may call the *Atiyah class with respect to* $[\gamma]$, vanishes, see also Atiyah's original work [2]. For the important particular case where the cocomplexes are either the smooth Lie group cohomology complexes of a Lie group H or the Chevalley-Eilenberg complexes of a Lie algebra $\mathfrak{h}$ with values in a short exact sequence of H-modules (resp. $\mathfrak{h}$-modules)

$$\{0\} \to P \to Q \to R \to \{0\}$$

the above problem is the *lifting of invariants* in the quotient module R to Q. For the particular Atiyah classes in the literature, the Lie algebra $\mathfrak{h}$ is a subalgebra of a bigger Lie algebra $\mathfrak{g}$, and $\mathfrak{u}$ is a second Lie algebra. We suppose that there is a either a Lie group U having Lie algebra $\mathfrak{u}$ and a Lie group homomorphism $\chi : H \to U$, or just a morphism of Lie algebras $\dot{\chi} : \mathfrak{h} \to \mathfrak{u}$. In the first case $\mathfrak{u}$ is a H-module (via $h\zeta = Ad_{\chi(h)}(\zeta)$), and in the second case $\mathfrak{u}$ is an $\mathfrak{h}$-module via $\zeta \mapsto [\dot{\chi}(\eta), \zeta]$ for all $\eta \in \mathfrak{h}$. Then the above short exact sequence of three H-modules (resp. $\mathfrak{h}$-modules) P, Q, and R specializes to the following:

$$\{0\} \to \mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{g}/\mathfrak{h}, \mathfrak{u}\big) \to \mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{g}, \mathfrak{u}\big) \to \mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{h}, \mathfrak{u}\big) \to \{0\}.$$

The class $[\gamma]$ in the zeroth cohomology group corresponding to $\mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{h}, \mathfrak{u}\big)$ is an invariant in $\mathbf{Hom}_{\mathbb{K}}\big(\mathfrak{h}, \mathfrak{u}\big)$, namely $T_e\chi$ (in the group case) or $\dot{\chi}$ (in the Lie algebra case). In most of the literature there is the following important particular case $U = GL(\mathfrak{g}/\mathfrak{h})$ or $\mathfrak{u} = \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}) = \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$, and $\chi(h) = Ad'_h$ (the induced adjoint representation in $\mathfrak{g}/\mathfrak{h}$ which is isomorphic to the tangent space of the homogeneous space G/H at the distinguished point $o = \pi(e)$ which is also called the linear isotropy representation, see [22, p.187]) or $ad'(\eta)$ (the analogous representation for the subalgebra $\mathfrak{h}$).

We shall give a review of these things which is meant -alas- pedagogical, and I am convinced that almost all the material presented here is more or less well-known, but some of it, as for instance the description of multi-differential operators or jet prolongations on homogeneous spaces is less easy to find in the literature, at least for me. In order to do this we shall formulate the underlying geometric concepts around not necessarily reductive homogeneous spaces in a mild categorical form. This is useful for the general programme relating homogeneous structures over a homogeneous space, such as vector bundles, fibre bundles, principal bundles, groupoids, etc. which mostly form a suitable category, to a very often small(er) category of the typical fibres which almost always can be expressed as an equivalence of categories. Moreover, to do at least a tiny bit of hopefully original work we shall generalize the G-invariant connections in a G-invariant principal bundle to those ones where the left G-action on the total space does no longer commute with the right action of the structure group U but is twisted by an automorphic action ϑ of G on U. The only example I found where this may be relevant is the treatment of most of the coadjoint orbits of the Poincaré group in geometric quantization: as time reversal is demanded to be antisymplectic by physicists, the connection (and hence its curvature form which is symplectic) is not fully invariant under the Poincaré group, but may change signs, whence the curvature form differs from the Kirillov-Kostant-Souriau form by a sign on some of the connected components of the orbit: this can be described by introducing the above automorphic action. I shall also describe how –on an infinitesimal (Lie algebra) level– coadjoint orbits can be generalized in the direction that given the Lie algebra $\mathfrak{g}$, given the Lie algebra of the structure group $\mathfrak{u}$, and given a linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ the subalgebra $\mathfrak{h}$ can be defined as an isotropy subalgebra of p in a certain way, see Proposition 2.9 in order to get infinitesimal G-ϑ-equivariant connections.

The second aim of these proceedings is to give a differential geometric interpretation –which I had announced a year ago– of the above-mentioned result [7] in terms of G-invariant symbol calculus on the homogeneous space because vanishing Atiyah class means to have a G-invariant linear connection in the tangent bundle of the homogeneous space.

At last I shall mention the result –that all experts in deformation quantization such as Nikolai will find completely unsurprising– that a coadjoint orbit having vanishing Atiyah class (for the tangent bundle) admits a G-invariant star-product which is clear by a result by B. Fedosov.

Notation: All manifolds are assumed to be smooth, Hausdorff and second countable. Following [23] we shall denote the category of all smooth manifolds whose morphism sets are smooth maps by $\mathcal{M}f$. In a cartesian product $M = M_1 \times \cdots \times M_n$ of sets let $\mathrm{pr}_i : M \to M_i$ denote the canonical projection on the ith factor. The symbol $\mathbb{K}$ will denote either the field of all real numbers or the field of all complex numbers. For a vector field X on a manifold M and a diffeomorphism $\Phi : M \to M$ let $\Phi^* X$ denote the pull-back $x \mapsto (T_x\Phi)^{-1}\big(X(\Phi(x))\big)$.

Acknowledgments: I'd like to thank the editorial team, in particular Martin Schlichenmaier for their infinite patience. Moreover I thank Damien Calaque for very useful discussions and several critical remarks. I also would like to thank Alberto Cattaneo for pointing out references [10] and [11], Ping Xu for pointing out reference [12], and Friedrich Wagemann for valuable help with the relation between Lie group and Lie algebra cohomology and for pointing out reference [34].

1 Some Preliminaries

In the first subsection 1.1 we just recall –in some homeopathical quantities of categorical language, see e.g. [27], [20]– well-known facts on fibered manifolds, principal bundles, associated bundles, and connections dealt with for instance in [21], [22], and [23] in order to fix notation. The second subsection 1.2 treats the notion of multi-differential operators on associated bundles which seems to be a bit less well-known.

1.1 Fibered Manifolds, Principal Bundles, their Associated Bundles, and Connections

Recall that a *fibered manifold* is a triple (E, τ, M) where E (the *total space*) and M (the *base*) are smooth manifolds and $\tau : E \to M$ (the *projection*) is a smooth surjective submersion. A morphism between two fibered manifolds (E, τ, M) and (E', τ', M') is a pair of smooth maps $\Phi : E \to E'$ and $\phi : M \to M'$ (the base map) such that the obvious diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ \tau\downarrow & & \downarrow\tau' \\ M & \xrightarrow{\phi} & M' \end{array} \tag{1.1}$$

commutes. The class of all fibered manifolds with their morphism sets hence forms a category denoted by $\mathcal{FM}$ by [23]. Returning to fibered manifolds, recall that –by the implicit function theorem– around any y_0 of the total space E of a fibered manifold (E, τ, M) there is a chart $\big(\mathcal{V}, \psi\big)$ and a chart $\big(\mathcal{U}, \varphi\big)$ around $\tau(y_0) \in M$ such that $\psi(\mathcal{V}) \subset \mathcal{U}$ and the local representative $\varphi \circ \tau|_{\mathcal{V}} \circ \psi^{-1}$ of τ is of the simple form $(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) \mapsto (x^1, \ldots, x^m)$. As a consequence, each fibre $\tau^{-1}(\{x\}) \subset E$ over $x \in M$ is a closed submanifold of E. Moreover, it follows that for any fibered manifold the differentiable structure of the base is uniquely determined by the differentiable structure on the total space and the condition that

the projection be a surjective submersion. Furthermore there is the well-known criterion of *passage to the quotient*: for any fibered manifold (E, τ, M) and any set-theoretic map $f : M \to M'$ such that $f \circ \tau : E \to M'$ is smooth it follows that f is smooth. In particular the base map of a morphism is uniquely induced by the map between the total spaces. Recall that a smooth *section* of a fibered manifold is a smooth map $\sigma : M \to E$ with $\tau \circ \sigma = \mathrm{id}_M$, and we shall denote the set of all smooth sections by $\Gamma^\infty(M, E)$. By the above-mentioned local form of τ the following well-known fact is clear that any fibered manifold (E, τ, M) admits *families of local sections*, $(\mathcal{U}_\kappa, \sigma_\kappa)_{\kappa \in \mathfrak{S}}$, i.e. there is an open cover $(\mathcal{U}_\kappa)_{\kappa \in \mathfrak{S}}$ of the base M and smooth maps $\sigma_\kappa : \mathcal{U}_\kappa \to \tau^{-1}(\mathcal{U}_\kappa)$ such that $\tau \circ \sigma_\kappa = \mathrm{id}_{\mathcal{U}_\kappa}$.
For a given manifold M we shall also need the subcategory $\mathcal{FM}_M$ of $\mathcal{FM}$ consisting of all *fibered manifolds over* M, i.e. where the objects all have the same base M, and all the base maps are equal to the identity map of M. We shall treat many more categories of more particular fibered manifolds, and for each such category $\mathcal{C}$ there will be an 'over M'-version denoted by $\mathcal{C}_M$ where all bases are equal to M and all base maps are equal to the identity map on M. Let $\iota_M : \mathcal{FM}_M \to \mathcal{FM}$ the inclusion functor. Since for any smooth section s of the fibered manifold (E, τ, M) and any morphism $(\Phi, \mathrm{id}_M) : (E, \tau, M) \to (E', \tau', M)$ of fibered manifolds over M the map $\Phi \circ s$ is a smooth section of (E', τ', M) it follows that the assignment $(E, \tau, M) \to \Gamma^\infty(M, E)$ and $\Phi \to \Gamma^\infty(M, \Phi) : (s \mapsto \Phi \circ s)$ defines a covariant functor $\Gamma^\infty(M, \ )$ from $\mathcal{FM}_M$ to Set.

Particular cases of fibered manifolds are the well-known *fibre bundles* which are given by quadruples (E, τ, M, S) such that (E, τ, M) is a fibered manifold, S is a manifold (the *typical fibre*), and such that there is an open cover $(\mathcal{U}_\kappa)_{\kappa \in \mathfrak{S}}$ of the base M and smooth isomorphisms $f_\kappa : (\tau^{-1}(\mathcal{U}_\kappa), \tau|_{\tau^{-1}(\mathcal{U}_\kappa)}, \mathcal{U}_\kappa) \to (\mathcal{U}_\kappa \times S, \mathrm{pr}_1, \mathcal{U}_\kappa)$ of fibered manifolds over $\mathcal{U}_\kappa$ (the *local trivialisations*). Hence each f_κ has the general form $f_\kappa(y) = \big(\tau(y), f_\kappa^{(2)}(y)\big)$ with a smooth map $f_\kappa^{(2)} : \tau^{-1}(\mathcal{U}_\kappa) \to S$. Note that for any chosen point $z_0 \in S$ the maps $\sigma_\kappa : \mathcal{U}_\kappa \to \tau^{-1}(\mathcal{U}_\kappa)$ given by $\sigma_\kappa(x) = f_\kappa^{-1}(x, z_0)$ are local sections. The class of all fibre bundles equipped with the morphism sets of its underlying fibered manifolds forms a category denoted by $\mathcal{FB}$ in [23]. For a given manifold M let $\mathcal{FB}_M$ denote the subcategory of all *fibre bundles over* M where all objects have the same base M, and all the base maps are equal to the identity map of M. In case the typical fibre is a finite-dimensional vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ there is the well-known (non full) subcategory $\mathbb{K}\mathcal{VB}$ of $\mathcal{FB}$ of all *vector bundles*: here every fibre carries the structure of a $\mathbb{K}$-vector space, and the morphisms have to be fibrewise $\mathbb{K}$-linear.
In particular for differential operators we shall encounter the following slightly more general situation: let $E := (E_n, \tau_n, M)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{K}$-vector bundles over M and for each $n \in \mathbb{N}$ let $i_n : E_n \to E_{n+1}$ be an injective vector bundle morphisms over M (projecting on the identity map on M). One is tempted to

form the "inductive limit $\lim_{n\to\infty} E_n$" which in general would no longer lead to a vector bundle over M with finite-dimensional fibres so it does not belong to the original category. However it is quite practical to consider such situations, e.g. the symmetric power of the tangent bundle, $\mathsf{S}(TM)$, which of course only symbolizes the sequence $\big(\oplus_{k=0}^{n} \mathsf{S}^k(TM)\big)_{n\in\mathbb{N}}$ of 'true' finite-dimensional vector bundles. We shall call these sequences *filtered vector bundles over* M, and agree upon that $\Gamma^\infty(M,E) := \lim_{n\to\infty} \Gamma^\infty(M,E_n)$ where the inductive limit is taken in the category of $\mathbb{K}$-vector spaces with respect to the linear maps of the section spaces induced by the $(i_n)_{n\in\mathbb{N}}$. Moreover a morphism of filtered vector bundles over M, $\Phi : E = (E_n, \tau_n, M, i_n)_{n\in\mathbb{N}} \to E' = (E'_n, \tau'_n, M, i'_n)_{n\in\mathbb{N}}$ is a sequence of vector bundle morphisms over M i.e. for each nonnegative integer n: $\Phi_n : E_n \to E'_n$ intertwining the maps i_n and i'_n, i.e. $\Phi_{n+1} \circ i_n = i'_n \circ \Phi_n$ for each $n \in \mathbb{N}$.

Let G be a Lie group where we write $e = e_G$ for its unit element and $(g_1, g_2) \mapsto g_1g_2$ for the multiplication. Let **Lie** $\mathcal{G}$ denote the category of all Lie groups where morphisms are smooth morphisms of Lie groups. Let $(\mathfrak{g}, [\ ,\ ])$ denote its Lie algebra. Recall that a *left G-space* (resp. *right G-space*) is a smooth manifold M equipped with a smooth *left G-action* $G \times M \to M$, mostly written $(g,x) \mapsto gx$, (resp. smooth *right G-action* $M \times G \to M$ mostly written $(x,g) \mapsto xg$) satisfying $g_1(g_2x) = (g_1g_2)x$ and $ex = x$ (resp. $(xg_1)g_2 = x(g_1g_2)$ and $xe = x$) for all $g_1, g_2 \in G$ and all $x \in M$. For each $x \in M$ let $\mathcal{O}_x^G$ denote the *G-orbit through x*, i.e. $\mathcal{O}_x^G = \{gx \in M \mid g \in G\}$ (resp. $\mathcal{O}_x^G = \{xg \in M \mid g \in G\}$ for right G-actions) which is of course well-known to be an immersed submanifold of M. Recall that an action is called *transitive* on M iff there is only one orbit. Recall that it is called *free* iff for all $g \in G$: if there is $x \in M$ such that $gx = x$ (resp. $xg = x$) then $g = e$. Moreover, for each $\xi \in \mathfrak{g}$ we shall denote the *fundamental vector field* of the left G-action (resp. right G-action) by $\xi_M(x) := \frac{d}{dt}\big(exp(t\xi)x\big)|_{t=0}$ (resp. $\xi^*(x) = \frac{d}{dt}\big(x(exp(t\xi))\big)|_{t=0}$) for all $x \in M$. Recall the Lie bracket rules $[\xi_M, \eta_M] = -[\xi,\eta]_M$ (resp. $[\xi^*, \eta^*] = [\xi,\eta]^*$) for all $\xi, \eta \in \mathfrak{g}$. Let G' be another Lie group, and M' a left G'-space (resp. a right G'-space). A pair (ϕ, θ) is called a morphism from the G-space M to the G'-space M' iff $\phi : M \to M'$ is a smooth map, and $\theta : G \to G'$ is a smooth morphism of Lie groups, such that the following intertwining property holds: $\phi(gx) = \theta(g)\phi(x)$ (resp. $\phi(xg) = \phi(x)\theta(g)$) for all $g \in G$ and $x \in M$ (we also say that ϕ is *G-θ equivariant*). Again the class of all left (resp. right) G-spaces with varying G forms a category. For fixed G we shall denote the subcategory of all left (resp. right) G-spaces whose morphisms all have $\theta = \mathrm{id}_G$ (so-called maps intertwining the G-action) by $G \cdot \mathcal{M}f$ (resp. $\mathcal{M}f \cdot G$). Finally note the functors $I : G \cdot \mathcal{M}f \to \mathcal{M}f \cdot G$ and $I : \mathcal{M}f \cdot G \to G \cdot \mathcal{M}f$ which replace actions by the action of the inverse, i.e. for a left G-space M one defines a right action by $yg := g^{-1}y$ for all $y \in M$ and $g \in G$. In this work we shall call left (resp. right) G-module a $\mathbb{K}$-vector space V ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) on which G acts from the left (resp. from the right) in a $\mathbb{K}$-linear way. As usual, V^G denotes the subspace of all fixed vectors (i.e. those $v \in V$ such that $gv = v$ for all $g \in G$).

Recall that for a fixed Lie group U a *principal bundle* over a manifold M with structure group U (or a principal U-bundle) is a fibre bundle (P, τ, M, U) equipped with a free right U-action $P \times U \to E$ such that for each $y \in P$ the fibre through y, $\tau^{-1}(\{\tau(y)\})$, coincides with the U-orbit $\{yu \in P \mid u \in G\}$ passing through y, and all the local trivializations $f_\kappa : \tau^{-1}(\mathcal{U}_\kappa) \to \mathcal{U}_\kappa \times U$ are U-equivariant in the sense that $f_\kappa^{(2)}(yu) = f_\kappa^{(2)}(y)u$ for all $y \in \tau^{-1}(\mathcal{U}_\kappa)$ and $u \in U$, see e.g. [21, p.50]. Note that any family of local sections $(\mathcal{U}_\kappa, \sigma_\kappa)_{\kappa \in \mathfrak{S}}$ of P ('local frames') gives rise to local trivializations via $f_\kappa^{-1} : \mathcal{U}_\kappa \times U \to \tau^{-1}(\mathcal{U}_\kappa)$ given by $f_\kappa^{-1}(x, u) = \sigma_\kappa(x)u$. It is not hard to see that the right U-action is always proper, see e.g. [32] for a definition. Conversely, by the slice theorem (see [32]) it follows that each right U-space P whose action is free and proper gives rise to a principal U-bundle $(P, \tau, P/U, U)$ over the quotient space $M = P/U$ of U-orbits where τ is the canonical projection. Note also that a fibered manifold (P, τ, M) admitting a smooth free right U-action on the total space P such that the fibres coincide with the U-orbits is automatically a principal U-bundle, see e.g. [23, p.87, Lemma 10.3]. Principal bundles (with varying U) form a category denoted by $\mathcal{PB}$ in [23] for which a morphism from a principal bundle (P, τ, M, U) to a principal bundle (P', τ', M', U') is a triple (Φ, θ, ϕ) where $(\Phi, \theta) : P \to P'$ is a morphism from the right U-space E to the right U'-space E' and $\phi : M \to M'$ is the map induced by Φ. Note that the forgetful functor $\mathcal{PB} \to \mathcal{FB}$ is not full, i.e. the above morphisms between principal bundles are more specific than just fibre-preserving maps. We shall denote by $\mathcal{PB}(U)$ the subcategory of all those principal bundles having fixed structure group U and morphisms as in $\mathcal{M}f \cdot U$, i.e. smooth maps $\Phi : P \to P'$ between total spaces intertwining the U-action, i.e. $\Phi(yu) = \Phi(y)u$ for all $y \in P$ and $u \in U$. We denote the subcategory of all principal fibre bundles over a fixed M having fixed structure group U by $\mathcal{PB}(U)_M$.

One particular case which is very important for us is the principal H-bundle $(G, \pi, G/H)$ where G is a Lie group, $H \subset G$ is a closed subgroup, and G/H is the quotient space of the right H-action on G given by right multiplication in the group G where $\pi : G \to G/H$ denotes the canonical projection. G/H is called a *homogeneous space* and is known to be a left G-space by means of the induced left multiplication $\ell : G \times G/H \to G/H$ given by $\ell(g', gH) = \ell_{g'}(gH) := (g'g)H$ in the group. This left G action is transitive.

Very frequently, we shall encounter *Lie algebra versions* of the preceding notions: a pair of (not necessarily finite-dimensional) Lie algebras $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h} \subset \mathfrak{g}$ is subalgebra can be called a *Lie algebra inclusion* (see e.g. [7]) or an *infinitesimal homogeneous space.*

Recall the important notion of an *associated bundle to a principal fibre bundle*: let (P, τ, M, U) be a principal U-bundle and S a left U-space. The right action $(P \times S) \times U \to P \times S$ given by $(y, z)u = (yu, u^{-1}z)$ is free and proper because the right U-action on P is free and proper. Then the quotient $P_U[S] = P[S] := (P \times S)/U$ is known to be a well-defined manifold where the canonical projection $P \times S \to P[S]$

is a smooth submersion. We shall not very often use the classical notation $P\varkappa_U S$ used for instance in [21] because it may be confused with fibered products. For computations we shall denote the equivalence class of the pair $(y,z)\in P\times S$ by $[y,z]\in P[S]$. The projection $\tau_{P[S]}: P[S]\to M$ given by $\tau_{P[S]}([y,z])=\tau(y)$ is a well-defined smooth surjective submersion, and the quadruple $(P[S],\tau_{P[S]},M,S)$ is a fibre bundle over M with typical fibre S, called the associated bundle to P: note that a family of local trivializations $(\mathcal{U}_\kappa, f_\kappa)_{\kappa\in\mathfrak{S}}$ for $P[S]$ can be obtained by a family of local sections $(\mathcal{U}_\kappa,\sigma_\kappa)_{\kappa\in\mathfrak{S}}$ of (P,τ,M,U) by setting $f_\kappa^{-1}(x,z)=[\sigma_\kappa(x),z]$. Moreover, note that any morphism $\phi: S\to S'$ of left U-spaces in $U\cdot\mathcal{M}f$ (i.e. of the particular form (ϕ,id_U)) gives rise to a well-defined morphism $P[\phi]: P[S]\to P[S']$ of fibre bundles over M given by $P[\phi]([y,z])=[y,\phi(z)]$. Hence the assignment $S\to P[S]$, $\phi\to P[\phi]$ defines a covariant functor, the *associated bundle functor* $P[\ ]$ from $U\cdot\mathcal{M}f$ to $\mathcal{FB}_M$.

We also need to recall the well-known description of *sections of associated bundles as U-equivariant maps*, see e.g. [21, p.115]: let S be a left U-space where the left U-action is denoted by l. Note first that for each y in the total space P of a principal U-bundle (P,τ,M,U) the smooth map $\Phi_y: S\to E[S]$ defined by $z\mapsto[y,z]$ is a diffeomorphism onto the fibre $\tau_{P[S]}^{-1}\big(\{\tau(y)\}\big)$ over $\tau(y)$, and clearly for each $u\in U$ $\Phi_{yu}=\Phi_y\circ l_u$. For any section $\sigma\in\Gamma^\infty(M,P[S])$ let $\hat\sigma: P\to S$ denote the map $y\mapsto\Phi_y^{-1}\Big(\sigma\big(\tau(y)\big)\Big)$ which is clearly well-defined, smooth and U-equivariant, i.e. $\hat\sigma(yu)=u^{-1}\hat\sigma(y)$. The map $\hat\sigma$ is called the *frame form of the section* σ, see e.g. [23, p.95]. Let $\mathcal{C}^\infty(P,S)^U$ denote the space of all U-equivariant smooth maps $P\to S$ which can be written as $\mathbf{Hom}_{\mathcal{M}f\cdot U}\big(F(P),I(S)\big)$ in categorical terms (where $F:\mathcal{PB}(U)\to\mathcal{M}f\cdot U$ denotes the forgetful functor). Conversely, let $f\in\mathcal{C}^\infty(P,S)^U$ and set $\check f: M\to P[S]$ as $\check f\big(\tau(y)\big)=[y,f(y)]$ which is clearly a well-defined smooth section of the fibre bundle $P[S]$ over M. The two maps $\hat{(\ )}$ and $\check{(\ )}$ are inverses and constitute a natural isomorphism of the covariant functors $S\to\Gamma^\infty(M,P[S])$ and $S\to\mathcal{C}^\infty(P,S)^U$ from $U\cdot\mathcal{M}f$ to Set.

Thirdly, recall for any smooth homomorphism of Lie groups $\varphi: U\to U'$ the associated bundle $P_U[U']$ where U acts on the left on U' via $u.u':=\varphi(u)u'$ carries a right U'-action defined by $[p,u']u'_1:=[p,u'u'_1]$ which is well-defined and free, and the fibres of $P_U[U']$ are clearly in bijection with the right U'-orbits. According to [23, p.87, Lemma 10.3], $P_U[U']$ is a principal U'-bundle over M. There is a natural morphism of principal bundles over M defined by

$$\Phi: P\to P_U[U']: p\mapsto[p,e_{U'}] \tag{1.2}$$

where the Lie group homomorphism $U\to U'$ is given by φ. Next, let U' act smoothly on the left of a smooth manifold S, and form the associated bundle $\big(P_U[U']\big)_{U'}[S]$. Since U also acts smoothly on the left on S via the action of U' and the homomorphism φ we can form the associated bundle $P_U[S]$ over M. It is not hard to see that the map (for all $p\in P$, $u'\in U'$, and $z\in S$)

$$\Phi_S: [p,z]\mapsto\big[[p,e_{U'}],z\big]\quad\text{with inverse}\quad\big[[p,u'],z\big]\mapsto[p,u'z] \tag{1.3}$$

is a well-defined smooth isomorphism $\Phi_S : P_U[S] \to \big(P_U[U']\big)_{U'}[S]$ of fibre bundles over M.

Finally, recall the notion of a *connection* in a principal fibre bundle (P, τ, M, U): let $(\mathfrak{u}, [\ ,\ ])$ denote the Lie algebra of the structure group U. We shall denote the right U-action on P by r. A connection 1-form α is a $\mathfrak{u}$-valued 1-form on the total space P, i.e. a smooth section in the vector bundle $\mathrm{Hom}(TP, \mathfrak{u})$ over P, such that

$$\forall\, y \in P,\ \forall\, u \in U: \quad \big(r_u^*\alpha\big)_y = \alpha_{yu} \circ T_y r_u \;=\; Ad_{u^{-1}} \circ \alpha_y, \tag{1.4}$$

$$\forall\, y \in P,\ \forall\, \zeta \in \mathfrak{u}: \quad \alpha_y(\zeta_y^*) \;=\; \zeta. \tag{1.5}$$

Let $(\Phi, \theta, \phi) : (P, \tau, M, U) \to (P', \tau', M', U')$ be a morphism between two principal fibre bundles, let α be a connection 1-form on (P, τ, M, U) and let α' be a connection 1-form on (P', τ', M', U'). Then both the pull-back $\Phi^*\alpha'$ and the form $T_e\theta \circ \alpha$ are $\mathfrak{u}'$-valued 1-forms over P. It is therefore reasonable to introduce the following *category of principal fibre bundles with connection*, written $\mathcal{PBC}$ whose objects are quintuples (P, τ, M, U, α) where (P, τ, M, U) is a principal fibre bundle equipped with a connection 1-form α, and where the set of morphisms is defined as follows

$$\begin{aligned} &\mathbf{Hom}_{\mathcal{PBC}}\big((P, \tau, M, U, \alpha), (P', \tau', M', U', \alpha')\big) := \\ &\big\{(\Phi, \theta, \phi) \in \mathbf{Hom}_{\mathcal{PB}}\big((P, \tau, M, U), (P', \tau', M', U')\big) \mid\ T_e\theta \circ \alpha = \Phi^*\alpha'\big\}. \end{aligned} \tag{1.6}$$

Again we can fix the structure group U to have the category $\mathcal{PBC}(U)$ where in the above definition of morphisms we set of course $\theta = \mathrm{id}_U$, and its 'over M'-version $\mathcal{PBC}(U)_M$. Recall that any principal bundle over a manifold M admits a connection, which can be seen by a partition of unity argument, see e.g. [21, p.67-68].
Moreover, recall that any connection in a principal U-bundle gives rise to a U-invariant splitting of the tangent bundle TP of the total space into the subbundle of vertical subspaces $\{\zeta^*(p) \mid \zeta \in \mathfrak{u}\}$, $p \in P$, and the bundle of horizontal subspaces $H_p := \{v_p \in T_pP \mid \alpha_p(v_p) = 0\}$. By means of this, one can introduce a *horizontal lift* of any vector field X on the base M to a vector field X^h on P which is uniquely determined by the condition that for each $p \in P$ the value $X^h(p)$ lies in H_p and $T_p\tau(X^h(p)) = X(\tau(p))$. A horizontal lift is always U-invariant: $r_u^*X^h = X^h$. Let $E = P[V]$ an associated vector bundle with H-module V. For any smooth section ψ of E let $\hat{\psi} \in \Gamma^\infty(P, V)^U$ be the corresponding U-equivariant map $P \to V$, and let X be a vector field on M. It is well-known that the following formula defines a *covariant derivative* $\nabla_X\psi$, i.e. a smooth section of E such that

$$\widehat{\nabla_X\psi} := X^h\big(\hat{\psi}\big), \tag{1.7}$$

see e.g. [21, p.116, Prop.1.3]. Clearly everything is well-defined since X^h is H-invariant. It follows that $(X, \psi) \mapsto \nabla_X\psi$ defines a bidifferential operator such that $\nabla_{fX}\psi = f\nabla_X\psi$ and $\nabla_X(f\psi) = X\big(f\big)\psi + f\nabla_X\psi$ for all vector fields X on

M, smooth real-valued functions f on M and smooth sections ψ of E: this is precisely the classical definition of a connection in a vector bundle, see e.g. [21, p.116, Prop.1.2].

1.2 Multidifferential Operators in Associated Vector Bundles

Let M be a manifold of dimension m, and let (E_1, τ_1, M),...,(E_k, τ_k, M), (F, τ_F, M) be $\mathbb{K}$-vector bundles over M of fibre dimension $p_1, \ldots, p_k, q$, respectively. There is an open cover $(\mathcal{U}_\kappa)_{\kappa\in\mathfrak{S}}$ of M trivializing all the $k+1$ vector bundles and serving as the family of domains for an atlas of M. For each integer $1 \leq j \leq k$ let $f_1^{(j)}, \ldots, f_{p_j}^{(j)}$ and $g_1, \ldots, g_q$ be local sections of E_j and F, respectively, forming a base of the free module of all local sections $\Gamma^\infty(\mathcal{U}_\kappa, E_j)$ and $\Gamma^\infty(\mathcal{U}_\kappa, F)$, respectively, over the ring $\mathcal{C}^\infty(\mathcal{U}_\kappa, \mathbb{K})$. Hence for each integer $1 \leq j \leq k$ any smooth section $\psi_{(j)}$ of $E_{(j)}$ and ψ of F is locally a linear combination $\psi_{(j)}|_{\mathcal{U}_\kappa} = \sum_{a_j=1}^{p_j} \psi_{(j)}^{a_j} f_{a_j}^{(j)}$ and $\psi|_{\mathcal{U}_\kappa} = \sum_{b=1}^{q} \psi^b g_b$, respectively, with smooth locally defined coefficient functions $\psi_{(j)}^{a_j}, \psi^b : \mathcal{U}_\kappa \to \mathbb{K}$. Furthermore, for any multi-index $I = (n_1, \ldots, n_m) \in \mathbb{N}^{\times m}$ let $|I| := n_1 + \cdots + n_m$ and let ∂_I be short for the partial derivative

$$\partial_I := \frac{\partial^{|I|}}{(\partial x^1)^{n_1} \cdots (\partial x^m)^{n_m}}.$$

Recall that a general *k-differential operator D of maximal order N* in the above bundles is a k-multilinear map $D : \Gamma^\infty(M, E_1) \times \cdots \times \Gamma^\infty(M, E_k) \to \Gamma^\infty(M, F)$ taking the following local form: there is a nonnegative integer N and for each collection of multi-indices $I_1, \ldots, I_k$ with $|I_j| \leq N$ (for all $1 \leq j \leq k$) and collection of positive integers $a_1, \ldots, a_k, b$ with $1 \leq a_j \leq p_j$ for all $1 \leq j \leq k$ and $1 \leq b \leq q$ there is a smooth $\mathbb{K}$-valued function $D_{\kappa a_1 \cdots a_k}^{b I_1 \cdots I_k}$ defined on $\mathcal{U}_\kappa$ such that for all $x \in \mathcal{U}_\kappa$ we get

$$\begin{aligned} (1.8)\quad & D\big(\psi_{(1)}, \ldots, \psi_{(k)}\big)(x) \\ &= \sum_{a_1=1}^{p_1} \cdots \sum_{a_k=1}^{p_k} \sum_{b=1}^{q} \sum_{\substack{I_1, \ldots, I_k \\ |I_1|, \ldots, |I_k| \leq N}} D_{\kappa a_1 \cdots a_k}^{b I_1 \cdots I_k}(x) \big(\partial_{I_1} \psi_{(1)}^{a_1}\big)(x) \cdots \big(\partial_{I_k} \psi_{(k)}^{a_k}\big)(x) g_b(x). \end{aligned}$$

The integer N, the maximal order, is of course required to be a global property of D and does not depend on the chart. The $\mathbb{K}$-vector space of all such k-differential operators of maximal order N will be denoted by the symbol $\mathbf{Diff}_M^{(N)}(E_1, \ldots, E_k; F)$. Let $\mathbf{Diff}_M(E_1, \ldots, E_k; F)$ the union of all the $\mathbf{Diff}_M^{(N)}(E_1, \ldots, E_k; F)$ in the $\mathbb{K}$-vector space of all k-multilinear maps. Recall that it is also a left module for the ring $\mathcal{C}^\infty(M, \mathbb{K})$ by multiplying smooth functions with the values of D. Recall the following $\mathbb{K}$-bilinear operadic composition $D' \circ_{j'} D$ of $D \in \mathbf{Diff}_M(E_1, \ldots, E_k; F)$ and $D' \in \mathbf{Diff}_M(F_1, \ldots, F_l; G)$ where $F_1, \ldots, F_l, G$ are also vector bundles over

M and there is an integer $1 \leq j_0 \leq l$ such that $F = F_{j_0}$: let $\psi_{(j)} \in \Gamma^\infty(M, E_j)$, $1 \leq j \leq k$, and $\chi_{(j')} \in \Gamma^\infty(M, F_{j'})$, $1 \leq j' \leq l$, $j' \neq j_0$ then

$$(1.9) \qquad (D' \circ_{j_0} D)\big(\chi_{(1)}, \ldots, \chi_{(j_0-1)}, \psi_{(1)}, \ldots, \psi_{(k)}, \chi_{(j_0+1)}, \ldots, \chi_{(l)}\big) := D'\Big(\chi_{(1)}, \ldots, \chi_{(j_0-1)}, D\big(\psi_{(1)}, \ldots, \psi_{(k)}\big), \chi_{(j_0+1)}, \ldots, \chi_{(l)}\Big)$$

is in $\mathbf{Diff}_M(F_1, \ldots, F_{j_0-1}, E_1, \ldots, E_k, F_{j_0+1}, \ldots, F_l; G)$, i.e. a $(k+l-1)$-differential operator. In the case $k = l = 1$ the above composition $\circ_1$ is course ordinary composition of linear maps.

Multidifferential operators can be seen as smooth sections of certain filtered vector bundles: we shall recall the notions of *jet bundles*, see the book [23], Section 12, for details. Let M and N be two manifolds having dimensions m and n, respectively, and let r be a nonnegative integer. Recall that two smooth curves $\gamma_1, \gamma_2 : \mathbb{R} \to M$ have rth order contact if in some (and a posteriori in any) chart φ the difference $\varphi \circ \gamma_1 - \varphi \circ \gamma_2$ vanishes to rth order at 0. Let $\phi, \psi : M \to N$ be two smooth maps. Recall that they are said to determine the same r-jet at $x \in M$ if for any smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$ the two smooth curves $\phi \circ \gamma$ and $\psi \circ \gamma$ have rth order contact at 0. Let ${}_xJ^r(M, N)$ denote the quotient of $\mathcal{C}^\infty(M, N)$ by the equivalence relation that $\phi \sim \psi$ iff ϕ and ψ determine the same r-jet at x, and let $J^r(M, N)$ be the disjoint union $\bigcup_{x \in M} {}_xJ^r(M, N)$. For each $\phi \in \mathcal{C}^\infty(M, N)$ let $j^r_x(\phi)$ denote its r-jet at x, i.e. the equivalence class of ϕ in ${}_xJ^r(M, N)$. In the particular case $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ the equivalence class j^r_x can be identified with the Taylor series of ϕ at x up to order r. There is an obvious surjective projection $\alpha : J^r(M, N) \to M$ by mapping $j^r_x(\phi)$ to x. Furthermore, the map $j^r_x(\phi) \to \phi(x)$ is a well-defined surjective projection $\beta : J^r(M, N) \to N$. Now each set $J^r(M, N)$ can be given a canonical differentiable structure of a smooth manifold of dimension $m + \binom{m+r}{r} n$ such that by means of the above projection $\pi^r_0 : J^r(M, N) \to M \times N : X \mapsto (\alpha(X), \beta(X))$ is a smooth fibre bundle over $M \times N$. For any $x' \in N$ let denote by $J^r(M, N)_{x'}$ the submanifold $\beta^{-1}(\{x'\})$ of $J^r(M, N)$. Note also that for two nonnegative integers r, s with $r \geq s$ there is a canonical surjective submersion $\pi^r_s : J^r(M, N) \to J^s(M, N)$ defined by $\pi^r_s\big(j^r_x(\phi)\big) = j^s_x(\phi)$. An important issue is the fact that for three manifolds M, N, P, each $x \in M$, and any two smooth maps $\phi : M \to P$ and $\psi : N \to P$ each r-jet $j^r_x(\psi \circ \phi)$ only depends on the r-jets $j^r_{\phi(x)}(\psi)$ and $j^r_x(\phi)$ which defines a composition $J^r(N, P) \times J^r(M, N) \to J^r(M, P)$ which is associative in the appropriate sense. Moreover, let (E, τ, M) be a $\mathbb{K}$-vector bundle over M. Define its *rth jet prolongation* to be the subset $J^rE := \{j^r_x(\phi) \in J^r(M, E) \mid x \in M \text{ and } \phi \in \Gamma^\infty(M, E)\}$ which is a smooth submanifold of $J^r(M, E)$. Defining linear combinations of r-jets of sections as the r-jet of the linear combination of sections endows the fibre bundle $(J^rE, \alpha|_{J^rE}, M)$ with the structure of a smooth vector bundle over M. In the same way it is shown that for the particular case of the trivial bundle $E = M \times \mathbb{K}$ every rth jet prolongation $J^r(M \times \mathbb{K}) = J^r(M, \mathbb{K})$ is a smooth $\mathbb{K}$-vector bundle over M whose fibres are associative commutative unital $\mathbb{K}$-algebras of dimension $\binom{m+r}{r}$ which are all isomorphic to the quotient of the free symmetric algebra $\mathsf{S}(\mathbb{K}^m)$ modulo the ideal $\oplus_{k=r+1}^{\infty}\mathsf{S}^k(\mathbb{K}^m)$. Furthermore, note that the multiplication of smooth sections by $\mathbb{K}$-valued smooth functions endows each J^rE with the structure of a (fibrewise) $J^r(M,\mathbb{K})$-module. There is the following canonical isomorphism of $\mathcal{C}^\infty(M,\mathbb{K})$-modules:

$$(1.10)\quad \Gamma^\infty\big(M, \mathbf{Hom}(J^rE_1\otimes\cdots\otimes J^rE_k, F)\big) \;\cong\; \mathbf{Diff}_M^{(r)}(E_1,\ldots,E_k;F)$$

defined by

$$(1.11)\qquad F \;\mapsto\; \Big((\psi_1,\ldots,\psi_k)\mapsto \Big(x\mapsto F_x\big(j_x^r(\psi_1)\otimes\cdots\otimes j_x^r(\psi_k)\big)\Big)\Big).$$

upon using the projections ${\pi^r_s}_j : J^rE_j \to J^sE_j$ for $r\geq s$ it is easy to see that the sequence of $\mathbb{K}$-vector bundles $\Big(\mathbf{Hom}(J^rE_1\otimes\cdots\otimes J^rE_k, F)\Big)_{r\in\mathbb{N}}$ is a filtered vector bundle with $i_r = ({\pi^{r+1}_r}_1\otimes\cdots\otimes{\pi^{r+1}_r}_k)^*$.

Let us consider now the well-known particular case where $M=G$ where G is a Lie group having Lie algebra $(\mathfrak{g},[\ ,\ ])$. Recall that for any $\xi\in\mathfrak{g}$ the fundamental field of the right multiplication $R: G\times G\to G$ where $R_g(g_0) := g_0g$ is given by the well-known *left-invariant vector field* denoted by $\xi^+(g) := T_eL_g(\xi)$ where $L: G\times G\to G$ is the canonical left multiplication $L_g(g_0) = gg_0$. Fix a base $e_1,\ldots,e_n$ of $\mathfrak{g}$. It is well-known that the tangent vectors $e_1^+(g),\ldots,e_n^+(g)$ form a vector space basis of T_gG for each $g\in G$. In the above formula (1.8) for the particular case $k=1$ and $E=F=G\times\mathbb{K}$ we can hence replace the iterated partial coordinate derivatives ∂_I by linear combinations with smooth coefficients of iterations of Lie derivatives with respect to left invariant vector fields.
These iterations correspond to algebraic iterations of Lie algebra elements described by the well-known *universal envelopping algebra* $\mathsf{U}(\mathfrak{g})$ *of the Lie algebra* $\mathfrak{g}$ (which we only need to consider over the field of real numbers, but which is of course much more general): $\mathsf{U}(\mathfrak{g})$ is defined to the quotient of the the free $\mathbb{R}$-algebra $\mathsf{T}(\mathfrak{g}) = \mathbb{R}1\oplus\bigoplus_{i=1}^\infty\mathsf{T}^i(\mathfrak{g})$ generated by the real vector space $\mathsf{T}^1(\mathfrak{g}) = \mathfrak{g}$ modulo the two-sided ideal I of $\mathsf{T}(\mathfrak{g})$ spanned by all elements of the form $a(\xi\eta-\eta\xi-[\xi,\eta])b$ with $a,b\in\mathsf{T}(\mathfrak{g})$ and $\xi,\eta\in\mathfrak{g}$. It is well-known that $\mathsf{U}(\mathfrak{g})$ has the structure of a real Hopf algebra: the counit map $\epsilon:\mathsf{U}(\mathfrak{g})\to\mathbb{R}$ sends $\lambda1\in\mathsf{U}(\mathfrak{g})$ to $\lambda\in\mathbb{R}$ and the image of $\bigoplus_{i=1}^\infty\mathsf{T}^i(\mathfrak{g})$ to zero, the comultiplication $\Delta:\mathsf{U}(\mathfrak{g})\to\mathsf{U}(\mathfrak{g})\otimes\mathsf{U}(\mathfrak{g})$ is defined to be as $\Delta(\xi) = \xi\otimes1+1\otimes\xi$ for all $\xi\in\mathfrak{g}$ and uniquely extends to a morphism of unital associative algebras $\mathsf{U}(\mathfrak{g})\to\mathsf{U}(\mathfrak{g})\otimes\mathsf{U}(\mathfrak{g})$, and the antipode $S:\mathsf{U}(\mathfrak{g})\to\mathsf{U}(\mathfrak{g})$ is defined by $S(\xi_1\cdots\xi_N) := (-1)^Nx_Nx_{N-1}\cdots x_2x_1$. The assignment $\mathfrak{g}\to\mathsf{U}(\mathfrak{g})$ is the left adjoint of the forgetful functor of the category of all real associative algebras to the category of all real Lie algebras, or in other words $\mathsf{U}(\mathfrak{g})$ is universal in the sense that each Lie algebra map $\phi:\mathfrak{g}\to A^-$ -where A^- is an associative algebra A seen as a Lie algebra with the commutator– uniquely lifts to a map of associative algebras $\Phi:\mathsf{U}(\mathfrak{g})\to A$ such that $\Phi\circ i_{\mathfrak{g}} = \phi$ where $i_{\mathfrak{g}}:\mathfrak{g}\to\mathsf{U}(\mathfrak{g})$ is the map induced by the natural injection $\mathfrak{g}\to\mathsf{T}(\mathfrak{g})$.

The Poincaré-Birkhoff-Witt-Theorem states that $\mathsf{U}(\mathfrak{g})$ is isomorphic to the vector space of the free commutative algebra generated by the vector space $\mathfrak{g}$, $\mathsf{S}(\mathfrak{g})$, as a cocommutative counital coalgebra. We shall write the comultiplication of an element $\mathsf{u} \in \mathsf{U}(\mathfrak{g})$ in Sweedler's notation $\Delta(\mathsf{u}) = \sum_{(\mathsf{u})} \mathsf{u}^{(1)} \otimes \mathsf{u}^{(2)}$. Note also that $\mathsf{U}(\mathfrak{g})$ is a filtered vector space, i.e. $\mathsf{U}(\mathfrak{g}) = \bigcup_{n\in\mathbb{N}} \mathsf{U}(\mathfrak{g})_n$ where each $\mathsf{U}(\mathfrak{g})_n$ is equal to $\oplus_{i=0}^{n} \mathsf{T}(\mathfrak{g})$ modulo I. Moreover the algebra $\mathsf{U}(\mathfrak{g})$ is filtered in the sense that $\mathsf{U}(\mathfrak{g})_n \mathsf{U}(\mathfrak{g})_p \subset \mathsf{U}(\mathfrak{g})_{n+p}$ for all nonnegative integers n, p, and that each $\mathsf{U}(\mathfrak{g})_n$ is a sub-coalgebra of the coalgebra $\mathsf{U}(\mathfrak{g})$.
Since the $\mathbb{R}$-linear map $\mathfrak{g} \to \mathbf{Diff}_G(G\times\mathbb{K}; G\times\mathbb{K})$ sending $\xi \in \mathfrak{g}$ to the Lie derivative with respect to the left invariant vector field ξ^+ is a morphism of Lie algebras we get a unique algebra map $\mathsf{U}(\mathfrak{g}) \to \mathbf{Diff}_G(G\times\mathbb{K}; G\times\mathbb{K})$ induced by the former and denoted by $\mathsf{u} \mapsto \mathsf{u}^+$. Consider now the $\mathbb{K}$-vector space $\mathcal{C}^\infty(G,\mathbb{K}) \otimes \mathsf{U}(\mathfrak{g})$ (here: $\otimes = \otimes_{\mathbb{R}}$) with unit $1 \otimes 1$ and multiplication given by (for all $\varphi, \psi \in \mathcal{C}^\infty(G,\mathbb{K})$ and for all $\mathsf{u}, \mathsf{v} \in \mathsf{U}(\mathfrak{g})$)

$$(\varphi \otimes \mathsf{u})(\psi \otimes \mathsf{v}) := \sum_{(\mathsf{u})} \varphi(\mathsf{u}^{(1)+}(\psi)) \otimes \mathsf{u}^{(2)}\mathsf{v}. \tag{1.12}$$

Moreover consider the following linear map $\mathcal{C}^\infty(G,\mathbb{K}) \otimes \mathsf{U}(\mathfrak{g}) \to \mathbf{Diff}_G(G\times\mathbb{K}; G\times\mathbb{K})$ defined by

$$(\varphi \otimes \mathsf{u}) \mapsto \Big(\psi \mapsto \varphi\big(\mathsf{u}^+(\psi)\big)\Big). \tag{1.13}$$

The following Proposition is well-known, see e.g. [17] for a star-product version, and not hard to check using the preceding facts.

Proposition 1.1. *The $\mathbb{K}$-vector space $\mathcal{C}^\infty(G,\mathbb{K}) \otimes \mathsf{U}(\mathfrak{g})$ equipped with its unit and multiplication (1.12) is an associative unital $\mathbb{K}$-algebra which is isomorphic as an associative unital $\mathbb{K}$-algebra to $\mathbf{Diff}_G(G\times\mathbb{K}; G\times\mathbb{K})$ equipped with the composition $\circ = \circ_1$ by means of the map (1.13).*

Let (P, τ, M, U) be a principal U-bundle, and let $V_1, \ldots, V_k, W$ be finite-dimensional vector spaces over $\mathbb{K}$ of dimension $p_1, \ldots, p_k, q$, respectively. Suppose that these vector spaces are left U-modules, i.e. U acts linearly on the left on each of these vector spaces where we denote the smooth linear action by $\rho_j : U \to GL(V_j)$ and $\rho : U \to GL(W)$, respectively, and by $\dot\rho_j : \mathfrak{u} \to \mathfrak{gl}(V_j)$ and $\dot\rho : \mathfrak{u} \to \mathfrak{gl}(W)$, respectively, the induced map of Lie algebras, i.e. $\dot\rho_j(\zeta) = \frac{d}{dt}\big(\rho_j(\exp(t\zeta))\big)\big|_{t=0}$. Let $E_1 := P[V_1], \ldots, E_k := P[V_k], F := P[W]$ the corresponding associated bundles over M which are of course $\mathbb{K}$-vector bundles. We should like to relate the space of k-differential operators on M, $\mathbf{Diff}_M(E_1, \ldots, E_k; F)$ to the corresponding space of k-differential operators on P, $\mathbf{Diff}_P(P\times V_1, \ldots, P\times V_k; P\times W)$: in the latter case, the bundles are trivial, hence the spaces of k-differential operators are easier to compute. It is clear that we can identify $\mathcal{C}^\infty(P, V)$ with $\Gamma^\infty(P, P\times V)$ for every finite-dimensional $\mathbb{K}$-vector space V. Recall also the natural isomorphism

$\Gamma^\infty(M,P[V]) \to \mathcal{C}^\infty(P,V)^U : \psi \mapsto \hat{\psi}$ for every finite-dimensional left U-module V which has been mentioned before. Each space of smooth functions $\mathcal{C}^\infty(P,V_j)$ $(1 \leq j \leq k)$ and $\mathcal{C}^\infty(P,W)$ is a U-module in the obvious way: for all $u \in U$ and $\psi'_{(j)} \in \mathcal{C}^\infty(P,V_j)$ one sets $u\psi'_{(j)} = \rho_j(u) \circ \psi'_{(j)} \circ r_u$ and likewise for $\psi' \in \mathcal{C}^\infty(P,W)$. There is an induced linear U-action on the space of k-differential operators, defined as usual for all $u \in U$ and $D' \in \mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$:

$$(1.14)\qquad (uD')\big(\psi'_{(1)},\ldots,\psi'_{(1)}\big) := \rho(u)\Big(D'\big(u^{-1}\psi'_{(1)},\ldots,u^{-1}\psi'_{(1)}\big)\Big).$$

Clearly, this action preserves operadic composition, i.e. for all $u \in U$ we have $u\big(D'_1 \circ_{j_0} D'_2\big) = (uD'_1) \circ_{j_0} (uD'_2)$. Moreover, for each integer $1 \leq j \leq k$ and each $\zeta \in \mathfrak{u}$ let $\zeta^* + \dot{\rho}_j(\zeta)$ denote the differential operator in $\mathbf{Diff}_P(P\times V_j;P\times V_j)$ given by the sum of the Lie derivative of the fundamental field (applied to the 'arguments' of a smooth function $\psi_{(j)} \in \mathcal{C}^\infty(P,V_j)$) and the linear map $\dot{\rho}_j(\zeta)$ (applied to the values of $\psi_{(j)}$). For each integer $1 \leq j \leq k$ let $\mathbf{K}_j$ be the subspace of all those k-differential operators in $\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$ which is spanned by all elements of the form

$$(1.15)\quad D\circ_j\big(\zeta^*+\dot{\rho}_j(\zeta)\big)\ \text{ where }\ D \in \mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W) \text{ and } \zeta \in \mathfrak{u}.$$

Since $u\big(\zeta^* + \dot{\rho}_j(\zeta)\big) = \big(Ad(u)(\zeta)\big)^* + \dot{\rho}_j\big(Ad(u)(\zeta)\big)$ for all $u \in U$ and $\zeta \in \mathfrak{u}$ it follows that each $\mathbf{K}_j$ is a U-submodule of the U-module of the above k-differential operators. Consider now the natural restriction map of $D' \in \mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$ to the U-equivariant sections $\hat{\psi}_{(1)} : P \to V_1$,..., $\hat{\psi}_{(k)} : P \to V_k$ which come from smooth sections $\psi_{(1)} \in \Gamma^\infty(M,P[V_1])$,..., $\psi_{(k)} \in \Gamma^\infty(M,P[V_k])$, so we define

$$(1.16)\qquad \begin{aligned} \mathrm{res} : \mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W) & \\ \to\ & \mathbf{Hom}_{\mathbb{K}}\big(\Gamma^\infty(M,P[V_1])\otimes\cdots\otimes\Gamma^\infty(M,P[V_1]);\mathcal{C}^\infty(P,W)\big) \\ D' \ \mapsto\ & \Big(\psi_{(1)}\otimes\cdots\otimes\psi_{(k)} \mapsto D'\big(\hat{\psi}_{(1)},\ldots,\hat{\psi}_{(k)}\big)\Big) \end{aligned}$$

Clearly $u\hat{\psi}_{(j)} = \hat{\psi}_{(j)}$ for all $u \in U$ and $1 \leq j \leq k$, hence each operator $\zeta^* + \dot{\rho}_j(\zeta)$ vanishes on each $\hat{\psi}_{(j)}$, hence

$$\mathrm{res}(\mathbf{K}_1 + \cdots + \mathbf{K}_k) = \{0\}$$

and the restriction map res passes to the quotient of the space all k-differential operators modulo $\mathbf{K}_1 + \cdots + \mathbf{K}_k$. There is the following

Proposition 1.2. *With the above notations:*
The following two vector spaces are isomorphic, viz

$$\frac{\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)^U}{\big(\mathbf{K}_1+\cdots+\mathbf{K}_k\big)^U} \cong \left(\frac{\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)}{\mathbf{K}_1+\cdots+\mathbf{K}_k}\right)^U$$

where the canonical injection of the left hand side into the right hand side is an isomorphism, and the restriction map maps the right hand side isomorphically to the space of all k-differential operators on M whence

$$\left(\frac{\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)}{\mathbf{K}_1+\cdots+\mathbf{K}_k}\right)^U \cong \mathbf{Diff}_M(E_1,\ldots,E_k;F).$$

Proof: Since we have to work locally we have to prepare the grounds by introducing some notation: for each integer $1\leq j\leq k$ let $v_1^{(j)},\ldots,v_{p_j}^{(j)}$ be a base of V_j, and let $w_1,\ldots,w_q$ be a base of W. Hence any smooth map $\psi'_{(j)}:P\to V_j$ and $\psi':P\to W$ is a linear combination $\sum_{a_j=1}^{p_j}\psi'^{a_j}_{(j)}v^{(j)}_{a_j}$ and $\sum_{b=1}^{q}\psi'^b w_b$ with smooth $\mathbb{K}$-valued coefficient functions. Next, choose a family of local sections $(\mathcal{U}_\kappa,\sigma_\kappa)_{\kappa\in\mathfrak{S}}$ of the principal bundle (P,τ,M,U) such that each $\mathcal{U}_\kappa$ is the domain of a chart of the manifold M. Hence for any smooth section $\psi_{(j)}$ of the bundle E_j and ψ of the bundle F the associated equivariant maps $\hat{\psi}_{(j)}:P\to V_j$ and $\hat{\psi}:P\to W$ are linear combinations with smooth coefficients of the above bases, and we therefore can compute the following particular local expressions for all $x\in\mathcal{U}_\kappa$:

$$\begin{aligned}
\psi_{(j)}(x) &= [\sigma_\kappa(x),\hat{\psi}_{(j)}\big(\sigma_\kappa(x)\big)] = \sum_{a_j=1}^{p_j}\hat{\psi}^{a_j}_{(j)}\big(\sigma_\kappa(x)\big)[\sigma_\kappa(x),v^{(j)}_{a_j}] =: \sum_{a_j=1}^{p_j}\psi^{a_j}_{(j)}(x)f^{(j)}_{a_j}(x),\\
\psi(x) &= [\sigma_\kappa(x),\hat{\psi}\big(\sigma_\kappa(x)\big)] = \sum_{b=1}^{q}\hat{\psi}^b\big(\sigma_\kappa(x)\big)[\sigma_\kappa(x),w_b] =: \sum_{b=1}^{q}\psi^b(x)g_b(x).
\end{aligned}$$

Hence we get the bijection for all $x\in\mathcal{U}_\kappa$ and $u\in U$

$$\psi^{a_j}_{(j)}(x) := \hat{\psi}^{a_j}_{(j)}\big(\sigma_\kappa(x)\big) \ \text{ and } \ \hat{\psi}^{a_j}_{(j)}\big(\sigma_\kappa(x)u\big) := \sum_{a'_j=1}^{p_j}\rho_j(u^{-1})^{a_j}_{a'_j}\psi^{a'_j}_{(j)}(x) \tag{1.17}$$

and likewise for ψ. Denote the coordinate vector field $\partial/(\partial x^\mu)$ in $\mathcal{U}_\kappa$ by ∂_μ for each integer $1\leq\mu\leq m=\dim(M)$. Then, in $\tau^{-1}(U_\kappa)\subset P$ define the following U-invariant horizontal lifts ∂^h_μ for all $x\in\mathcal{U}_\kappa$ and $u\in U$:

$$\partial^h_\mu\big(\sigma_\kappa(x)u\big) := T_{\sigma_\kappa(x)}r_u\big(T_x\sigma_\kappa(\partial_\mu)\big).$$

Clearly, these horizontal lifts are U-invariant, commute, are τ-related with the ∂_μ, and commute with all the fundamental fields ζ^*, $\zeta\in\mathfrak{u}$. For each multi-index $I\in\mathbb{N}^m$ denote by ∂^h_I the iteration $(\partial^h_1)^{n_1}\cdots(\partial^h_m)^{n_m}$. Let $e_1,\ldots,e_n$ be a vector space base of the Lie algebra $\mathfrak{u}$. For each multi-index $J=(n'_1,\ldots,n'_n)\in\mathbb{N}^{\times n}$ let e_J denote the product $e_1^{n'_1}\cdots e_n^{n'_n}$ in the universal envelopping algebra $\mathsf{U}(\mathfrak{u})$ of $\mathfrak{u}$. For each $\mathsf{u}\in\mathsf{U}(\mathfrak{u})$ let $\mathsf{u}\mapsto\mathsf{u}^*$ denote the algebra map of $\mathsf{U}(\mathfrak{u})$ into the differential operators on $\mathcal{C}^\infty(P,\mathbb{K})$ induced by the Lie algebra map $\zeta\mapsto\zeta^*$ from the Lie algebra $\mathfrak{u}$ to the fundamental fields being part of the differential operators. Since obviously the vector fields $\partial^h_1, \ldots, \partial^h_m$, $e^*_1 \ldots, e^*_n$ are a local base of all the vector fields in $\tau^{-1}(\mathcal{U}_\kappa)$ any differential operator D' in $\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$ takes the following local form for all $y\in\tau^{-1}(\mathcal{U}_\kappa)$

$$(1.18)\qquad D'\big(\psi'_{(1)},\ldots,\psi'_{(k)}\big)(y) = \sum_{a_1=1}^{p_1}\cdots\sum_{a_k=1}^{p_k}\sum_{b=1}^{q}\sum_{\substack{I_1,\ldots,I_k\\|I_1|,\ldots,|I_k|\leq N}}\sum_{\substack{J_1,\ldots,J_k\\|J_1|,\ldots,|J_k|\leq N}} D'^{bI_1\cdots I_kJ_1\cdots J_k}_{\kappa a_1\cdots a_k}(y) \big(e^*_{J_1}\partial^h_{I_1}\psi'^{a_1}_{(1)}\big)(y)\cdots\big(e^*_{J_k}\partial^h_{I_k}\psi'^{a_k}_{(k)}\big)(y)\; w_b.$$

Inserting the U-equivariant maps $\hat{\psi}_{(j)}: P\to V_j$ we get for all integers $1\leq\mu\leq m$ and $1\leq\nu\leq n$ using eqn (1.17)

$$(1.19)\qquad \partial^h_\mu(\hat{\psi}^{a_j}_{(j)}) = \widehat{\partial_\mu(\psi^{a_j}_{(j)})} \quad\text{and}\quad e^*_\nu(\hat{\psi}^{a_j}_{(j)}) = -\sum_{a_j=1}^{p_j}\dot{\rho}_j(e_\nu)^{a_j}_{a'_j}\hat{\psi}^{a'_j}_{(j)}.$$

This shows already that if D' was U-equivariant modulo $\mathbf{K} := \mathbf{K}_1+\cdots+\mathbf{K}_k$, i.e. for each $u\in U$ there is $K_u\in\mathbf{K}$ such that $uD' = D'+K_u$ then $\mathrm{res}(D')$ maps $(\psi_{(1)},\ldots,\psi_{(k)})$ to a U-equivariant smooth map $P\to W$ which can be identified with a smooth section in $\Gamma^\infty(M,P[W])$. The above local considerations show that $\mathrm{res}(D')$ is k-differential on M.
Let us show that the kernel of the restriction map is in $\mathbf{K}_1+\cdots+\mathbf{K}_k$ (the other inclusion has already been shown): Let $D'\in\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)$ such that the restriction of D' to any k U-equivariant maps vanishes. Looking at is local form (1.18) we can transform all the derivatives with respect to fundamental fields into matrix-multiplication using the equations (1.19) and conclude that the modified local operator

$$\check{D}'_\kappa\big(\psi'_{(1)},\ldots,\psi'_{(k)}\big)(y) = \sum_{a_1,a'_1=1}^{p_1}\cdots\sum_{a_k,a'_k=1}^{p_k}\sum_{b=1}^{q}\sum_{\substack{I_1,\ldots,I_k\\|I_1|,\ldots,|I_k|\leq N}}\sum_{\substack{J_1,\ldots,J_k\\|J_1|,\ldots,|J_k|\leq N}} D'^{bI_1\cdots I_kJ_1\cdots J_k}_{a_1\cdots a_k}(y) \dot{\rho}_1\big(S(e_{J_1})\big)^{a_1}_{a'_1}\big(\partial^h_{I_1}\psi'^{a'_1}_{(1)}\big)(y)\cdots\dot{\rho}_k\big(S(e_{J_1})\big)^{a_k}_{a'_k}\big(\partial^h_{I_k}\psi'^{a'_k}_{(k)}\big)(y)\; w_b$$

always vanishes on *all* smooth functions $\psi'_{(1)},\ldots,\psi'_{(k)}$ on $\tau^{-1}(\mathcal{U}_\kappa)$ having values in V_1,..., V_k, respectively: indeed, since $\tau^{-1}(\mathcal{U}_\kappa)$ is diffeomorphic to $\mathcal{U}_\kappa\times U$ it is clear that $\check{D}'_\kappa$ only contains derivatives in the direction of $\mathcal{U}_\kappa$, and can hence be considered as a family of k-differential operators on $\mathcal{U}_\kappa$ parametrised by U. By the local form of the U-equivariant sections $\hat{\psi}_{(j)}$, (1.17), we see that the functions $\psi^{a'_j}_{(j)}$ are completely arbitrary. Therefore the 'family' vanishes, hence $\check{D}'_\kappa$ vanishes. We can thus subtract $\check{D}'_\kappa$ from D' without changing D'. In this difference the following terms will occur

$$\begin{aligned}
&\delta^{a_1}_{a'_1}\big(e^*_{J_1}\partial^h_{I_1}\psi'^{a'_1}_{(1)}\big)(y)\cdots\delta^{a_k}_{a'_k}\big(e^*_{J_k}\partial^h_{I_k}\psi'^{a'_k}_{(k)}\big)(y)\\
&\qquad -\dot\rho_1\big(S(e_{J_1})\big)^{a_1}_{a'_1}\partial^h_{I_1}\psi'^{a'_1}_{(1)}\big)(y)\cdots\dot\rho_k\big(S(e_{J_k})\big)^{a_k}_{a'_k}\partial^h_{I_k}\psi'^{a'_k}_{(k)}\big)(y)\\
&= \sum_{r=1}^{k}\Big(\delta^{a_1}_{a'_1}\big(e^*_{J_1}\partial^h_{I_1}\psi'^{a'_1}_{(1)}\big)(y)\cdots\delta^{a_{r-1}}_{a'_{r-1}}\big(e^*_{J_{r-1}}\partial^h_{I_{r-1}}\psi'^{a'_{r-1}}_{(r-1)}\big)(y)\\
&\qquad\qquad\Big(\delta^{a_r}_{a'_r}\big(e^*_{J_r}\partial^h_{I_r}\psi'^{a'_r}_{(r)}\big)(y)-\dot\rho_r\big(S(e_{J_r})\big)^{a_r}_{a'_r}\partial^h_{I_r}\psi'^{a'_r}_{(r)}\big)(y)\Big)\\
&\qquad\qquad\dot\rho_{r+1}\big(S(e_{J_{r+1}})\big)^{a_{r+1}}_{a'_{r+1}}\partial^h_{I_{r+1}}\psi'^{a'_{r+1}}_{(r+1)}\big)(y)\cdots\dot\rho_k\big(S(e_{J_k})\big)^{a_k}_{a'_k}\partial^h_{I_k}\psi'^{a'_k}_{(k)}\big)(y)\Big)\,,
\end{aligned}$$

and the difference in the rth summand always contains a factor of $\zeta^*\mathrm{id}_{V_r}-\dot\rho_r(\zeta)$ in its matrix form: indeed, according to the Poincaré-Birkhoff-Witt-Theorem, the vector space $\mathsf{U}(\mathfrak{u})$ is spanned by the monomials ζ^N, $\zeta\in\mathfrak{u}$, and N any nonnegative integer, and we get for all $N\geq 1$

$$(\zeta^N)^*\mathrm{id}_{V_r}-\dot\rho_r(\zeta^N)=\sum_{t=0}^{N-1}(\zeta^t)^*\dot\rho_r(\zeta^{N-1-t})\Big(\zeta^*\mathrm{id}_{V_r}-\dot\rho_r(\zeta)\Big).$$

It follows that for each $\kappa\in\mathfrak{S}$ there are locally defined differential operators $D'_{\kappa 1},\ldots,D'_{\kappa k}$ in $\tau^{-1}(\mathcal{U}_\kappa)$ such that $D'_{\kappa j}\in\mathbf{K}_j$ for all $1\leq j\leq k$ and $D'=D'_{\kappa 1}+\cdots+D'_{\kappa k}$ locally on $\tau^{-1}(\mathcal{U}_\kappa)$. Let $(\chi_\kappa)_{\kappa\in\mathfrak{S}}$ be a partition of unity subordinate to the open cover $(\mathcal{U}_\kappa)_{\kappa\in\mathfrak{S}}$. Defining the global differential operator $D'_j=\sum_{\kappa\in\mathfrak{S}}(\chi_\kappa\circ\tau)D'_{\kappa j}$ for each $1\leq j\leq k$ we see that $D'_j\in\mathbf{K}_j$ and $D'=D'_1+\cdots+D'_k$, showing

$$\mathrm{Ker}(\mathrm{res})=\mathbf{K}_1+\cdots+\mathbf{K}_k.$$

Next, let D be any k-differential operator in $\mathbf{Diff}_M(E_1,\ldots,E_k;F)$ given locally as in equation (1.8) where we use the particular base sections $f^{(j)}_{a_j}(x)=[\sigma_\kappa(x),v^{(j)}_{a_j}]$, $1\leq j\leq k$, $1\leq a_j\leq p_j$, for the bundles E_j, and $g_b(x)=[\sigma_\kappa(x),w_b]$, $1\leq b\leq q$ for the bundle F. Define for all $x\in\mathcal{U}_\kappa$ and $u\in U$ and each combination of indices the smooth map

$$D'^{bI_1\cdots I_k}_{\kappa a_1\cdots a_k}\big(\sigma_\kappa(x)u\big):=\sum_{a'_1=1}^{p_1}\cdots\sum_{a'_k=1}^{p_k}\sum_{b'=1}^{q}\rho(u^{-1})^b_{b'}D^{b'I_1\cdots I_k}_{\kappa a'_1\cdots a'_k}(x)\rho_1(u)^{a'_1}_{a_1}\cdots\rho_k(u)^{a'_k}_{a_k}.$$

Now form the local k-differential operator D'_κ as in eqn (1.18) where the multi-indices $J_1,\ldots,J_k$ are void, and globalize the expression to a k-differential operator $D'=\sum_{\kappa\in\mathfrak{S}}(\chi_\kappa\circ\tau)D'_\kappa$ on P by using the above partition of unity $(\chi_\kappa)_{\kappa\in\mathfrak{S}}$. It can easily be checked that D' is U-equivariant and that the restriction to U-equivariant smooth maps gives back D which proves surjectivity of the restriction map to the k-differential operators. We have the isomorphism

$$\frac{\mathbf{Diff}_P(P\times V_1,\ldots,P\times V_k;P\times W)^U}{\big(\mathbf{K}_1+\cdots+\mathbf{K}_k\big)^U}\cong\mathbf{Diff}_M(E_1,\ldots,E_k;F).$$

and since the k-differential operators on P which are only U-equivariant modulo $\mathbf{K}$ restrict to k-differential operators on M, the other stated isomorphism is also clear. □

2 G-ϑ-equivariant Principal Bundles with Connections over Homogeneous Spaces

In this Section we shall mainly be interested in G-equivariant structures over *homogeneous spaces*: let G be a Lie group having Lie algebra $(\mathfrak{g}, [\ ,\ ])$, let $H \subset G$ be a closed subgroup where $\mathfrak{h} \subset \mathfrak{g}$ denotes its Lie algebra. Let $M = G/H$ denote the homogeneous space where $\pi : G \to G/H$ is the canonical projection, and $o = \pi(e) \in M$ the distinguished point. Recall that $(G, \pi, G/H, H)$ is a principal H-bundle over G/H.

2.1 Some G-equivariant Versions of Categories of Fibered Manifolds

Let G be a Lie group with Lie algebra $(\mathfrak{g}, [\ ,\ ])$. All the categories mentioned in the previous Section can now be considered in a 'G-equivariant way': Let $G \cdot \mathcal{FM}$ be the category of all *G-equivariant fibered manifolds*: the class of objects consists of all those fibered manifolds (E, τ, M) where the total space E and the base M are both left G-spaces and the projection $\tau : E \to M$ is G-equivariant, i.e. $\tau(gy) = g\tau(y)$ for all $g \in G$ and $y \in E$, and each set of morphisms consists of all those morphisms $\Phi : (E, \tau, M) \to (E', \tau', M')$ of fibered manifolds which in addition intertwine the left G-actions, i.e. $\Phi(gy) = g\Phi(y)$ for all $g \in G$ and all $y \in E$. In the same way we define the category of all *G-equivariant fibre bundles*, $G \cdot \mathcal{FB}$, and the category of all *G-equivariant vector bundles*, $G \cdot \mathcal{VB}$. Note also also the G-equivariant versions 'over M', i.e. the categories $G \cdot \mathcal{FM}_M$, $G \cdot \mathcal{FB}_M$, and $G \cdot \mathcal{VB}_M$.

For a G-equivariant version of the category $\mathcal{PB}(U)$ of all principal fibre bundles with fixed structure group U we first have to say how the left G-action and the right U-action on the total space P of a principal bundle (P, τ, M, U) are related: the simplest choice would be to declare that they commute, i.e. $g(pu) = (gp)u$ for all $g \in G$, $u \in U$, $p \in P$. In this case we shall speak of the category of all *G-equivariant principal bundles with fixed structure group U* (where morphisms intertwine all the actions) denoted by $G \cdot \mathcal{PB}(U)$, and its 'over M' version $G \cdot \mathcal{PB}(U)_M$. In the next Section we shall, however, treat the slightly more general version: let $\vartheta : G \times U \to U$ be a smooth left *automorphic* G-action on the Lie group U: that is, ϑ is a smooth left G-action on the manifold U such that for each $g \in G$ the map $\vartheta_g : U \to U : u \mapsto \vartheta(g, u)$ is an *Lie group automorphism of* U, i.e.

for all $u_1, u_2 \in U$

$$\vartheta_g(u_1u_2) = \vartheta_g(u_1)\vartheta_g(u_2) \quad \text{and} \quad \vartheta_g(e_U) = e_U. \tag{2.1}$$

Note that if U modulo its identity component is a finitely generated group then the group $\mathrm{Aut}(U)$ of all Lie group automorphisms $U \to U$ carrying the compact-open topology is itself a Lie group, see e.g. [18] in which case the map $g \mapsto \vartheta_g$ is a smooth Lie group homomorphism $G \to \mathrm{Aut}(U)$. But we shall not need this restriction.

A principal U-bundle (P, τ, M, U) over a left G-space M having structure group U is called *G-ϑ-equivariant* iff there is a left G-action ℓ' on the total space P – which we shall mostly write $\ell'_g(p) = gp$ for all $g \in G$ and $p \in P$- projecting on the left G-action ℓ on M such that

$$\forall\ g \in G,\ p \in P,\ u \in U: \qquad g(pu) = (gp)\vartheta_g(u). \tag{2.2}$$

We shall denote the corresponding category by $G \cdot \mathcal{PB}(U;\vartheta)$ (where morphisms simply intertwine all the group actions) and its 'over M' version by $G\cdot\mathcal{PB}(U;\vartheta)_M$. For the trivial case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$ we would return to the aforementioned *G-equivariant principal U-bundles (over M)*, $G \cdot \mathcal{PB}(U)$ and $G \cdot \mathcal{PB}(U)_M$.

Finally, a G-equivariant version of principal U-bundles with connection can be obtained as follows: objects are quintuples (P, τ, M, U, α) where the principal U-bundle is G-ϑ-equivariant, and $\alpha \in \Gamma^\infty(P, T^*P \otimes \mathfrak{u})$ is a connection 1-form satisfying

$$\forall\ g \in G: \quad \ell'^*_g\alpha = T_{e_U}\vartheta_g \circ \alpha. \tag{2.3}$$

Morphisms in this category between (P, τ, M, U, α) and $(P', \tau', M', U, \alpha')$ are just smooth maps $\Phi : P \to P'$ of total spaces intertwining the left G- and the right U-action (thereby inducing a unique G-equivariant smooth map $\phi : M \to M'$ on the bases) such that $\Phi^*\alpha' = \alpha$. We shall denote this category by $G \cdot \mathcal{PBC}(U, \vartheta)$ (and its 'over M'-version by $G \cdot \mathcal{PBC}(U, \vartheta)_M$), and in the particular case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$ it will be denoted by $G \cdot \mathcal{PBC}(U)$ (with 'over M-version' $G \cdot \mathcal{PBC}(U)_M$).

An example with nontrivial ϑ will be given in Subsubsection 2.5.2.

2.2 G-equivariant Fibered manifolds over Homogeneous Spaces

Let $M = G/H$ be a homogeneous space. Note first that for any left G-space E any smooth G-equivariant map $f : E \to M$ is automatically a surjective submersion: let $x_0 \in M$ a value of f, i.e. $x_0 = f(y_0)$ for some $y_0 \in E$. Then for all $g \in G$ one has $gx_0 = gf(y_0) = f(gy_0)$ showing surjectivity because of the transitivity of the G-action on M. Furthermore, any tangent vector v at $x \in M$ is the value of a fundamental field $\xi_M(x)$ of the left G-action on M. Let $y \in E$ such that $f(y) = x$ then $T_y f\big(\xi_E(y)\big) = \xi_M\big(f(y)\big) = v$ showing that f is a submersion.

Let $(E, \tau, G/H)$ be a G-equivariant fibered manifold over M. There is an obvious functor $G \cdot \mathcal{FM}_M \to H \cdot \mathcal{M}f$ assigning to $(E, \tau, G/H)$ the fibre E_o over the distinguished point $o = \pi(e)$ (which is a left H-space since H fixes o). Any G-equivariant smooth map of total spaces inducing the identity map on M induces an H-equivariant map on the fibres over o. On the other hand, using the fact that $(G, \pi, G/H, H)$ is a G-equivariant principal H-bundle over G/H, we see that for any left H-space S the associated bundle functor $S \to G_H[S]$ which maps the morphism f of left H-spaces $S \to S'$ to the map $[g, z] \mapsto [g, f(z)]$ of associated fibre bundles over M. The following Proposition –which seems to be well-known– shows that the two functors constitute an equivalence of categories:

Proposition 2.1. *Let $G \cdot \mathcal{FM}_M$ the category of G-equivariant fibered manifolds over $M = G/H$ and $H \cdot \mathcal{M}f$ the category of smooth left H-spaces. Then the two functors $G_H[\;] : H \cdot \mathcal{M}f \to G \cdot \mathcal{FM}_M$ and $(\;)_o : G \cdot \mathcal{FM}_M \to H \cdot \mathcal{M}f$ constitute an equivalence of categories*

$$H \cdot \mathcal{M}f \;\; \simeq \;\; G \cdot \mathcal{FM}_{G/H}$$

Proof: Consider first $(\;)_o \circ G_H[\;]$. For any left H-space S let $\psi_S : G_H[S]_o \to S$ be the inverse of the map $\Phi_o : S \to G_H[S]_o$ given by $\Phi_o(z) = [e, z]$ which has been mentioned earlier. It is easy to check that $S \to \psi_S$ constitute a natural isomorphism $(\;)_o \circ G_H[\;]$ to $\mathrm{id}_{H \cdot \mathcal{M}f}$. On the other hand, let (E, τ, M) be in $G \cdot \mathcal{FM}_M$. There is a canonical smooth map $\Phi_{(E,\tau,M)} = \Phi : G_H[E_o] \to E$ sending $[g, z]$ to gz which clearly induces the identity on M and is natural in (E, τ, M). It is easy to check that the map Φ is a bijection. We shall show that the map $\hat{\Phi} : G \times E_o \to E$ defined by $\hat{\Phi}(g, z) = gz$ is a submersion which shows that Φ is also a submersion, and since $\dim(G_H[E_o]) = \dim(E)$ Φ is a local diffeomorphism and therefore a diffeomorphism being bijective: indeed, let let $y \in E$ and $v \in T_y E$. Writing $x = \tau(y) \in M$ we have $w := T_y \tau(v) \in T_x M$, and since M is homogeneous there is $g \in G$ and $\xi \in \mathfrak{g}$ such that $x = go$ and $w = \frac{d}{dt}(g\exp(t\xi)o)|_{t=0}$. Let $z := g^{-1} y \in E_o$, and consider

$$v' := \frac{d}{dt}(g\exp(t\xi)z)|_{t=0} = T_{(g,z)}\hat{\Phi}\big(T_e L_g(\xi), 0\big) \in T_y E.$$

Clearly $T_y\tau(v') = w = T_y\tau(v)$, so $v - v' \in \mathrm{Ker}\; T_y\tau = T_y(E_x)$ since τ is a submersion. Writing $\ell_g : M \to M$ for the left G-action on M and $\ell'_g : E \to E$ for the left G-action on E, set $v'' := (T_z\ell'_g)^{-1}(v - v') \in T_z E$. Clearly, by equivariance of τ we get that $v'' \in \mathrm{Ker}\; T_z\tau = T_z E_o$. Therefore $v = v' + T_z\ell'_g(v'')$, and we get

$$v = T_{(g,z)}\hat{\Phi}\big(T_e L_g(\xi), v''\big),$$

whence $\hat{\Phi}$ is a submersion, which ends the proof. □

Note that this implies that every G-equivariant (weakly) fibered manifold over M is isomorphic to a G-equivariant fibre bundle whence the two categories $G\cdot\mathcal{FM}_M$ and $G\cdot\mathcal{FB}_M$ are also equivalent.

Recall that the space of all smooth sections $\Gamma^\infty(G/H, G_H[S])$ is in bijection with the space of all H-equivariant functions $\mathcal{C}^\infty(G,S)^H$. It is easy to see that the space of all G-invariant smooth sections is isomorphic to the subset of fixed points of the H-action, i.e.

$$S^H \cong \Gamma^\infty(G/H, G_H[S])^G \tag{2.4}$$

where the isomorphism is given by $z \mapsto \big(\pi(g)\mapsto [g,z]\big)$.

In a similar manner it is shown that the category of *left H-modules* (which are finite-dimensional $\mathbb{K}$ vector spaces) is equivalent to the category of all *G-equivariant vector bundles over M*, $G\cdot\mathcal{VB}_M$.

Recall that the *tangent bundle* of G/H is isomorphic to

$$G_H[\mathfrak{g}/\mathfrak{h}] \cong TM \tag{2.5}$$

where the Lie group H acts on the quotient $\mathfrak{g}/\mathfrak{h}$ as follows: let $\varpi:\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ be the canonical projection, then the following representation $h\mapsto Ad'_h$ on $\mathfrak{g}/\mathfrak{h}$ is well-defined

$$Ad'_h\big(\varpi(\xi)\big) \quad := \quad \varpi\big(Ad_h(\xi)\big) \tag{2.6}$$

since $\mathfrak{h}$ is stable by all the Ad_h, $h\in H$. Note that the kernel of the linear map $T_e\pi : T_eG = \mathfrak{g} \to T_o(G/H)$ is equal to $\mathfrak{h}$, hence there is the linear isomorphism $\pi' : \mathfrak{g}/\mathfrak{h} \to T_o(G/H)$ which is clearly H-equivariant with respect to the actions (2.6) and $h\mapsto T_o\ell_h$. The above mentioned isomorphism of vector bundles is given by

$$[g,z] \mapsto T_e\ell_g\big(\pi'(z)\big) \tag{2.7}$$

for all $g\in G$ and $z\in\mathfrak{g}/\mathfrak{h}$. Let us compute the *Lie bracket of two vector fields X and Y on G/H*: identifying $T(G/H)$ with the associated bundle $G_H[\mathfrak{g}/\mathfrak{h}]$ there are the two frame forms $\hat{X},\hat{Y} : G\to\mathfrak{g}/\mathfrak{h}$, i.e. smooth H-equivariant functions corresponding to the two sections X, Y. Now fix a connection 1-form α in the principal H-bundle $G\to G/H$. In general α is NOT G-invariant, see the next Chapter. Let $\tilde{X},\tilde{Y}\in\Gamma^\infty(G,TG)$ be the horizontal lifts with respect to α, see the end of Subsection 1.1. Then the pairs $(\tilde{X},X)$ and $(\tilde{Y},Y)$ are π-related, i.e. $T\pi\circ\tilde{X} = X\circ\pi$ and $T\pi\circ\tilde{Y} = Y\circ\pi$, and $\tilde{X}$ and$\tilde{Y}$ are invariant under the right multiplication with H. Define the smooth functions $\hat{X}',\hat{Y}' : G\to\mathfrak{g}$ by $\hat{X}'(g) = (T_eL_g)^{-1}\big(\tilde{X}(g)\big)$ and $\hat{Y}'(g) = (T_eL_g)^{-1}\big(\tilde{Y}(g)\big)$. Then both $\hat{X}'$ and $\hat{Y}'$ are H-equivariant, i.e. $\hat{X}'(gh) = Ad_{h^{-1}}\big(\hat{X}'(g)\big)$ and likewise for $\hat{Y}'$ for all $g\in G$ and $h\in H$, and project to the frame forms, i.e. $\varpi\circ\hat{X}' = \hat{X}$ and $\varpi\circ\hat{Y}' = \hat{Y}$. Since the pairs $(\tilde{X}, X)$ and $(\tilde{Y}, Y)$ are π-related, the same holds for the pair $\big([\tilde{X}, \tilde{Y}], [X, Y]\big)$ which allows to compute the Lie bracket of X and Y upon using the frame forms:

$$(2.8)\quad [X, Y]_{\mathrm{Lie}}(\pi(g)) = \Big[g, \hat{X}'(g)^{+}\big(\hat{Y}\big)(g) - \hat{Y}'(g)^{+}\big(\hat{X}\big)(g) + \varpi\big([\hat{X}'(g), \hat{Y}'(g)]\big)\Big]$$

It is straight forward to check that the above formula is well-defined and does not depend on the connection chosen.

Exercise: Show that the frame form of the fundamental field $\xi_{G/H}$ of the left G-action ℓ on G/H is given by (for all $g \in G$ and $\xi \in \mathfrak{g}$)

$$\widehat{\xi_{G/H}}(g) = \varpi\big(Ad_{g^{-1}}(\xi)\big).$$

Exercise: Let U be a Lie group, and $\theta : U \to G$, $j : H \to U$ be smooth Lie group homomorphisms such that the following diagram commutes

$$\begin{array}{ccccc} G & & \stackrel{\theta}{\longleftarrow} & & U \\ & i \nwarrow & & \nearrow j & \\ & & H & & \end{array}$$

where $i : H \to G$ is the natural inclusion of subgroups. Show that the associated bundle $G_H[U/j(H)]$ over G/H carries the structure of a G-equivariant *Lie groupoid* over the unit space G/H, see [26] or [28] for definitions, where (for all $g, g_1, g_2 \in G$, $u, u_1, u_2 \in U$) the target projection t equals the bundle projection, the source projection s is given by $s([g, u \bmod j(H)]) = \pi(g\theta(u))$, the unit map is given by $1(\pi(g)) = [g, e_U \bmod j(H)]$, the multiplication by

$$\mu\Big([g_1, u_1 \bmod j(H)], [g_2, u_2 \bmod j(H)]\Big) = \big[g_1, u_1 j\big(\theta(u_1)^{-1} g_1^{-1} g_2\big) u_2 \bmod j(H)\big],$$

and the inverse by $[g, u \bmod j(H)]^{-1} = [g\theta(u), u^{-1} \bmod j(H)]$. Moreover show that every G-equivariant Lie groupoid is isomorphic (in that category) to a Lie groupoid of the above form (equivalence of appropriate categories). Hint: define the Lie group U as the pull-back of the principal bundle $(G, \pi, G/H, H)$ over G/H to the t-fibre E_o over the distinguished point o by means of the restriction of the source projection s to E_o.

2.3 G-ϑ-equivariant Principal Bundles over Homogeneous Spaces

Let G be a Lie group with Lie algebra $(\mathfrak{g}, [\ ,\ ])$, let $H \subset G$ be a closed subgroup with Lie algebra $(\mathfrak{g}, [\ ,\ ])$, and let M be the homogeneous space $M = G/H$. Let U be a Lie group with Lie algebra $(\mathfrak{u}, [\ ,\ ])$, and let $\vartheta : G \times U \to U$ be an automorphic left G-action on U, written $\vartheta(g, u) = \vartheta_g(u)$.

For the following it is rather convenient to form the *semidirect product* $G_\vartheta \times U$ of the two Lie groups U and G with respect to the automorphic action ϑ: recall that the underlying manifold is $G \times U$, and for all $u, u_1, u_2 \in U$ and $g, g_1, g_2 \in G$ the multiplication is defined by

$$(g_1, u_1)(g_2, u_2) = \big(g_1 g_2, \vartheta_{g_2^{-1}}(u_1)u_2\big) \tag{2.9}$$

whence the unit element is (e, e_U), and the inverse of (g, u) is given by $\big(g^{-1}, \vartheta_g(u^{-1})\big)$. We shall denote left and right multiplication in $G_\vartheta \times U$ by L^ϑ and R^ϑ, respectively, i.e. $L^\vartheta_{(g_1,u_1)}(g_2, u_2) = (g_1, u_1)(g_2, u_2) = R^\vartheta_{(g_2,u_2)}(g_1, u_1)$ for all $g_1, g_2 \in G$ and $u_1, u_2 \in U$. Note that we have chosen a less common convention for the semidirect product in order to maintain the order $G \times U$ as opposed to the usual $U \times G$. Moreover recall that the subset $\{e\} \times U$ is a closed normal subgroup of $G_\vartheta \times U$ with factor group isomorphic to G. A concrete isomorphism is realized by the projection $\mathrm{pr}_1 : G_\vartheta \times U \to G$.

Now let $(P, \tau, G/H, U)$ a G-ϑ-equivariant principal U-bundle over G/H. In order to get an idea about the relevant structures involved, consider its fibre $P_o = \tau^{-1}(\{o\})$ over the distinguished point $o = \pi(e) \in G/H$, and *choose an element* $y_P \in P_o$. Since the map $U \to P_o$ given by $u \mapsto y_P u$ is a diffeomorphism, and since P_o is a *left* H-space there is a unique smooth map $\check{\chi}_P : H \to U$ such that for all $h \in H$:

$$y_P \check{\chi}_P(h) := h y_P. \tag{2.10}$$

We clearly have $\check{\chi}_P(e) = e_U$, and we get for all $h_1, h_2 \in H$

$$\begin{aligned} y_P \check{\chi}_P(h_1 h_2) &= (h_1 h_2) y_P = h_1 (h_2 y_P) = h_1 \big(y_P \check{\chi}_P(h_2)\big) \overset{(2.2)}{=} (h_1 y_P)\vartheta_{h_1}\big(\check{\chi}_P(h_2)\big) \\ &= \big(y_P \check{\chi}_P(h_1)\big)\vartheta_{h_1}\big(\check{\chi}_P(h_2)\big) = y_P\big(\check{\chi}_P(h_1)\vartheta_{h_1}\big(\check{\chi}_P(h_2)\big)\big) \end{aligned}$$

Hence $\check{\chi}_P(h_1 h_2) = \check{\chi}_P(h_1)\vartheta_{h_1}\big(\check{\chi}_P(h_2)\big)$, and the map

$$\chi : H \to U : h \mapsto \vartheta_{h^{-1}}\big(\check{\chi}_P(h)\big) \tag{2.11}$$

is easily checked to satisfy the identity

$$\forall\ h_1, h_2 \in H : \qquad \chi(h_1 h_2) = \vartheta_{h_2^{-1}}\big(\chi(h_1)\big)\chi(h_2). \tag{2.12}$$

Recall that the preceding equation (2.12) is the defining condition for the smooth map χ to be a *crossed homomorphism (with respect to* ϑ*)* $H \to U$. In the trivial case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$ the map χ is a smooth homomorphism of Lie groups. Moreover note that the constant map $\chi(h) = e_U$ for all $h \in H$ is always a crossed homomorphism w.r.t. ϑ. Furthermore, recall that a smooth map $\chi : H \to U$ is a crossed homomorphism w.r.t. ϑ if and only if the combined map

$$\tilde{\chi} : H \to G\ {}_\vartheta \times U : h \mapsto \tilde{\chi}(h) := \big(h, \chi(h)\big)$$

is a homomorphism of Lie groups, i.e. $\tilde{\chi}(e) = (e, e_U)$ and $\tilde{\chi}(h_1 h_2) = \tilde{\chi}(h_1)\tilde{\chi}(h_2)$ for all $h_1, h_2 \in H$, and this is in turn equivalent to the fact that $\chi = \mathrm{pr}_2 \circ \tilde{\chi}$ for some Lie group homomorphism $\tilde{\chi} : H \to G\,{}_\vartheta\!\times U$ satisfying $\mathrm{pr}_1 \circ \tilde{\chi} = i_H$ (where $i_H : H \to G$ denotes the natural inclusion).

Define the following set:

$$\mathcal{P} = \mathcal{P}_\vartheta(H,U) := \{\chi : H \to U \mid \chi \text{ crossed homomorphism w.r.t. } \vartheta\}. \tag{2.13}$$

In order to make this set into the set of all objects of a small category we note that for any $u \in U$ and $h \in H$ the map $h \mapsto (e,u)\tilde{\chi}(h)(e,u)^{-1} =: \tilde{\chi}'(h)$ is again a morphism of Lie groups $H \to G\,{}_\vartheta\!\times U$ such that $\mathrm{pr}_1 \circ \tilde{\chi}' = i_K$, whence for any $u \in U$ and $h \in H$ the map $u.\chi : h \mapsto \vartheta_{h^{-1}}(u)\chi(h)u^{-1}$ is again a crossed homomorphism $H \to U$. It is not hard to see that the map $U \times \mathcal{P}_\vartheta(H,U) \to \mathcal{P}_\vartheta(H,U)$ is a left H-action (in the sense of sets). We define the morphism sets as follows: for each $\chi, \chi' \in \mathcal{P}$

$$\mathbf{Hom}_{\mathcal{P}}(\chi,\chi') := \{u \in U \mid (u.\chi)(h) = \vartheta_{h^{-1}}(u)\chi(h)u^{-1} = \chi'(h)\ \forall\ h \in H\}, \tag{2.14}$$

where composition of morphisms is defined by group multiplication in U. It immediately follows that each morphism is an isomorphism whence the small category $\mathcal{P}_\vartheta(H,U)$ is a groupoid, in fact the action groupoid of the above left U-action on $\mathcal{P}_\vartheta(H,U)$.

In order to define an associated bundle $G_H[U]$ with typical fibre U we unfortunately need to modify the semidirect product $G_\vartheta \times U$ by a diffeomorphism to relate the multiplication in this product to the convention of the right H-action for associated bundles: let $\Xi : G_\vartheta \times U \to G \times U$ the diffeomorphism

$$\Xi(g,u) := (g, u^{-1}) \tag{2.15}$$

which is an involution on the underlying manifold $G \times U$, i.e. $\Xi \circ \Xi = \mathrm{id}_{G\times U}$. Upon using left and right multiplications in the semidirect product define the following group actions on $G \times U$: $\hat{L} : G \times (G \times U) \to G \times U$, $\hat{R} : (G \times U) \times U \to G \times U$, $\hat{P} : U \times (G \times U) \to G \times U$, and $R^\chi : (G \times U) \times H \to G \times U$ where $\hat{L}$ will be a left G-action, $\hat{R}$ will be a right U-action, $\hat{P}$ will be a left U-action, and R^χ will be a right H-action: for all $g, g_0 \in G$, $h \in H$, and $u, u_0, \tilde{u} \in U$:

$$g_0(g,u) := \hat{L}_{g_0}(g,u) := (g_0 g, u) = \big(\Xi^{-1} \circ L^\vartheta_{(g_0,e_U)} \circ \Xi\big)(g,u), \tag{2.16}$$
$$(g,u)u_0 := \hat{R}_{u_0}(g,u) := \big(g, u\vartheta_{g^{-1}}(u_0)\big) = \big(\Xi^{-1} \circ L^\vartheta_{(e,u_0^{-1})} \circ \Xi\big)(g,u) \tag{2.17}$$
$$\tilde{u}(g,u) := \hat{P}_{\tilde{u}}(g,u) := (g, \tilde{u}u) = \big(\Xi^{-1} \circ R^\vartheta_{(e,\tilde{u}^{-1})} \circ \Xi\big)(g,u) \tag{2.18}$$
$$R^\chi_{h_0}(g,u) := \big(gh_0, \chi(h_0)^{-1}\vartheta_{h_0^{-1}}(u)\big) = \big(\Xi^{-1} \circ R^\vartheta_{\tilde{\chi}(h)} \circ \Xi\big)(g,u) \tag{2.19}$$

All these actions are well-defined, and the definition of R^χ (2.19) shows that the map $\lambda^\chi : H \times U \to U$ defined by

$$\lambda^\chi_{h_0}(u) = \lambda^\chi(h_0,u) := \chi(h_0^{-1})^{-1}\vartheta_{h_0}(u) = \vartheta_{h_0}\big(\big(\chi(h_0)u\big)\big) \tag{2.20}$$

is a smooth left H-action on U such that $R^{\chi}_{h_0}(g,u) = \big(gh_0, \lambda^{\chi}_{h_0^{-1}}(u)\big)$. Since $\hat{L}$ and $\hat{R}$ come from left multiplications in the semidirect product, whereas $\hat{P}$ and R^{χ} come from a right multiplications it follows that all the maps $\hat{L}_{g_0}$ and $\hat{R}_{u_0}$ commute with all the maps $\hat{P}_{\tilde{u}}$ and $R^{\chi}_{h_0}$. Moreover we get

$$g_0\big((g,u)u_0\big) = \big(g_0g, \vartheta_{g^{-1}}(u_0)\big) = (g_0g,u)\vartheta_{g_0}(u_0) \tag{2.21}$$

$$\hat{P}_{\tilde{u}}\big(R^{\chi}_{h_0}(g,u)\big) = \big(gh_0, \tilde{u}\vartheta_{h_0^{-1}}\big(\chi(h_0^{-1})u\big)\big) = R^{\tilde{u}.\chi}_{h_0}\big(\hat{P}_{\tilde{u}}(g,u)\big) \tag{2.22}$$

Note that the subgroup H of G becomes a closed subgroup $\tilde{\chi}(H)$ of the semidirect product $G_{\vartheta} \times U$ via $\tilde{\chi}$: let $(h_n)_{n\in\mathbb{N}}$ be a sequence in H such that the sequence $\big(\tilde{\chi}(h_n)\big)|_{n\in\mathbb{N}}$ converges. In particular its first component $(h_n)_{n\in\mathbb{N}}$ converges to $h \in H$, hence $\big(\tilde{\chi}(h_n)\big)|_{n\in\mathbb{N}}$ converges to $\tilde{\chi}(h) \in \tilde{\chi}(H)$.

We can now define a functor $\mathbf{P} : \mathcal{P}_{\vartheta}(H,U) \to G \cdot \mathcal{PB}(U;\vartheta)_{G/H}$ as follows: for each $\chi \in \mathcal{P}_{\vartheta}(H,U)$ let P_{χ} be the associated bundle $G_H[U]$ where the subgroup H acts on U on the left via λ^{χ}, see (2.20). By definition of the right H-action (2.19) this is equal to $(G\times U)/H$. Let

$$\kappa_{\chi} = \kappa : G \times U \to P_{\chi} = G_H[U] \tag{2.23}$$

denote the projection, and for any $(g,u) \in G\times U$ we shall write $[g,u]_{\chi}$ for $\kappa(g,u) \in P_{\chi}$. Since H acts freely and properly on the right on $G\times U$ via R^{χ} (because Ξ is a diffeomorphism and $\tilde{\chi}(H)$ is a closed subgroup of $G_{\vartheta}\times U$) the above left G-action $\hat{L}$ (2.16) and the above right U-action $\hat{R}$ (2.17) pass to the quotient P_{χ} to define a left G-action ℓ' and a right U-action r such that κ intertwines the actions, i.e. $\ell'_{g_0} \circ \kappa = \kappa \circ \hat{L}_{g_0}$ and $r_{u_0} \circ \kappa = \kappa \circ \hat{R}_{u_0}$ for all $g_0 \in G$ and $u_0 \in U$. We get for all $g \in G$ and $u \in U$:

$$\ell'_{g_0}\big([g,u]\chi\big) = g_0[g,u]_{\chi} := [g_0g,u]_{\chi}, \tag{2.24}$$

$$r_{u_0}\big([g,u]_{\chi}\big) = [g,u]_{\chi}u_0 := [g,u\vartheta_{g^{-1}}(u_0)]_{\chi}. \tag{2.25}$$

Equation (2.21) passes to the quotient as follows:

$$g_0\big(\big([g,u]_{\chi}\big)u_0\big) = \big([g_0g,u]_{\chi}\big)\vartheta_{g_0}(u_0) \tag{2.26}$$

Note that the right U-action on P_{χ} is free: if for some $g \in G$, $u, u_0 \in U$ we have $[g,u]_{\chi} = [g,u]_{\chi}u_0 = [g,u\vartheta_{g^{-1}}(u_0)]_{\chi}$ it follows that $u = u\vartheta_{g^{-1}}(u_0)$ hence $u_0 = e_U$ whence the action is free. Moreover the right U-orbits coincide with the fibres of the associated bundle: by definition, the right U-orbits are contained in the fibres, on the other hand each $[g,u]_{\chi} \in \tau^{-1}\big(\pi(g)\big)$ is equal to $[g,e_U]_{\chi}\vartheta_g(u)$, hence in the right U-orbit passing through $[g,e_U]_{\chi}$. Using [23, p.87, Lemma 10.3] it follows that $(P_{\chi},\tau,G/H,U)$ is a principal U-bundle over G/H which is G-ϑ-equivariant by eq (2.26). Next, let $\chi' \in \mathcal{P}_{\vartheta}(H,U)$, and let $\tilde{u} \in \mathbf{Hom}_{\mathcal{P}}(\chi,\chi')$ whence $\chi' = \tilde{u}.\chi$.

Using eqn (2.22) we see that the left U-action $\hat{P}_{\tilde{u}}$ on $G \times U$ induces a unique map $P_{\tilde{u}} : P_\chi \to P_{\tilde{u}.\chi}$ such that $P_{\tilde{u}} \circ \kappa_\chi = \kappa_{\tilde{u}.\chi} \circ \hat{P}_{\tilde{u}}$ for all $\tilde{u} \in U$. We get

$$P_{\tilde{u}}\big([g,u]_\chi\big) := [g, \tilde{u}u]_{\tilde{u}.\chi} \tag{2.27}$$

for all $g \in G$ and $u, \tilde{u} \in U$. Since $\hat{P}_{\tilde{u}}$ commutes with all $\hat{L}_{g_0}$ and $\hat{R}_u$ it follows that $P_{\tilde{u}}$ is a morphism of G-equivariant principal U-bundles over G/H. Clearly $P_{e_U} = \mathrm{id}_{P_\chi}$ and $P_{\tilde{u}_1} \circ P_{\tilde{u}_2} = P_{\tilde{u}_1\tilde{u}_2}$ for all $\tilde{u}_1, \tilde{u}_2 \in U$, whence $\chi \to P_\chi$, $\tilde{u} \mapsto P_{\tilde{u}}$ is a covariant functor $\mathcal{P}_\vartheta(H,U) \to G \cdot \mathcal{PB}(U;\vartheta)_{G/H}$.

Conversely, in order to construct a functor $\mathbf{X} : G \cdot \mathcal{PB}(U;\vartheta)_{G/H} \to \mathcal{P}_\vartheta(H,U)$ let $(P, \tau, G/H, U)$ a G-ϑ-equivariant principal U-bundle over G/H, consider its fibre $P_o = \tau^{-1}(\{o\})$ over the distinguished point $o = \pi(e) \in G/H$, and *choose an element* $y_P \in P_o$. By the preceding considerations there is a unique crossed homomorphism $\chi_P : H \to U$ such that $y_P \vartheta_h\big(\chi_P(h)\big) := hy_P$ for all $h \in H$. Hence we get an assignment $\mathbf{X} : G \cdot \mathcal{PB}(U;\vartheta)_{G/H} \to \mathcal{P}_\vartheta(H,U)$ where $(P, \tau, G/H, U)$ is assigned the crossed homomorphism χ_P. Again note that the assignment depends on the choice $y_P \in P_o$. Furthermore, let $\Phi : (P, \tau, G/H, U) \to (P', \tau', G/H, U)$ a morphism of G-ϑ-equivariant principal U-bundles over G/H. Then $\Phi(y_P) \in P'_o$ whence there is a unique $\tilde{u}_\Phi \in U$ such that $\Phi(y_P) = y_{P'}\tilde{u}_\Phi$. We compute for all $h \in H$

$$\begin{aligned} \Phi(hy_P) &= \Phi\big(y_P \check{\chi}_P(h)\big) = \Phi(y_P)\check{\chi}_P(h) = (y_{P'}\tilde{u}_\Phi)\check{\chi}_P(h) = y_{P'}\big(\tilde{u}_\Phi \check{\chi}_P(h)\big) \\ \Phi(hy_P) &= h\Phi(y_P) = h\big(y_{P'}\tilde{u}_\Phi\big) = (hy_{P'})\vartheta_h(\tilde{u}_\Phi) = y_{P'}\big(\check{\chi}_{P'}(h)\vartheta_h(\tilde{u}_\Phi)\big) \end{aligned}$$

Applying $\vartheta_{h^{-1}}$ to the resulting equation for $\check{\chi}_P$ and $\check{\chi}_{P'}$ we get

$$\chi_{P'}(h) = \vartheta_{h^{-1}}(\tilde{u}_\Phi)\chi_P(h)\tilde{u}_\Phi^{-1}$$

whence $\tilde{u}_\Phi \in \mathbf{Hom}_{\mathcal{P}}(\chi_P, \chi_{P'})$. It is easily checked that the rule assigning to the bundle $(P, \tau, G/H, U)$ the crossed homomorphism χ_P and to the morphism Φ the group element $\tilde{u}_\Phi$ is a covariant functor $\mathbf{X} : G \cdot \mathcal{PB}(U;\vartheta)_{G/H} \to \mathcal{P}_\vartheta(H,U)$.

Proposition 2.2. *The two functors* $\mathbf{P} : \chi \to P_\chi$ *and* $\mathbf{X} : P \to \chi_P$ *constitute an equivalence of the small category* $\mathcal{P}_\vartheta(H,U)$ *and the large category* $G \cdot \mathcal{PB}(U;\vartheta)_{G/H}$*,*

$$\mathcal{P}_\vartheta(H,U) \quad \simeq \quad G \cdot \mathcal{PB}(U;\vartheta)_{G/H}.$$

Proof: The above considerations show that the two functors are well-defined. If we choose for each $\chi \in \mathcal{P}_\vartheta(H,U)$ the element $y_{P_\chi} = [e, e_U]_\chi \in (P_\chi)_o$ it is easy to check that the composite functor $\mathbf{X} \circ \mathbf{P}$ equals the identity functor $\mathcal{P}_\vartheta(H,U) \to \mathcal{P}_\vartheta(H,U)$. For the composition $\mathbf{P} \circ \mathbf{X}$ define the map $\Phi_P : (\mathbf{P} \circ \mathbf{X})(P) \to P$ by

$$\Phi_P([g,u]_{\chi_P}) := g(y_P u). \tag{2.28}$$

It is easy to check using the right H-action (2.19), the definition of χ_P (2.10), and the proof of Proposition 2.1 that it is a well-defined isomorphism of G-ϑ-equivariant principal U-bundles over G/H. For a morphism $\Psi : P \to P'$ in $G \cdot \mathcal{PB}(U;\vartheta)_{G/H}$ we use the formula $(\mathbf{P} \circ \mathbf{X})(\Psi) = P_{\tilde{u}_\Psi}$ to show that $P \to \Phi_P$ is a natural isomorphism. □

2.4 G-ϑ-equivariant connections and Atiyah classes

In this Section we should like to define small category which will be equivalent to $G \cdot \mathcal{PBC}(U)_{G/H}$.

2.4.1 A slight generalization of Wang's Theorem

In order to get an idea, we fix a crossed homomorphism $\chi : H \to U$ and consider the associated principal U-bundle $(P_\chi, \tau, G/H, U)$ of the preceding Subsection 2.3. It will be much more convenient to work on the manifold $G \times U$ and use the projection $\kappa_\chi : G \times U \to P_\chi$. However, as it turns out, the semidirect product $G_\vartheta \times U$ will be even better thanks to its structure of a Lie group. Let $\tilde{\kappa} = \tilde{\kappa}_\chi : G_\vartheta \times U \to P_\chi$ be the canonical projection (recall that $P_\chi = (G_\vartheta \times U)/\tilde{\chi}(H))$, and we get the relation between the two projections via the involution Ξ see (2.15)

$$\kappa_\chi \circ \Xi = \tilde{\kappa}_\chi \tag{2.29}$$

In any of the two cases $G \times U$ or $G_\vartheta \times U$, we have a principal H-bundle over P_χ. Recall the notion of a *tensorial* 1*-form* $\hat{\alpha}$ (resp. $\tilde{\alpha}$) with values in $\mathfrak{u}$ on $G \times U$ (resp. $G_\vartheta \times U$), see e.g. [21, p.75]: for any $\eta \in \mathfrak{h}$ let $\hat{\eta}^*$ (resp. $\tilde{\eta}^*$) the fundamental vector field $\hat{\eta}^*(g,u) = \frac{d}{dt}\big(R^\chi_{\exp(t\eta)}(g,u)\big)|_{t=0}$ (resp. $\tilde{\eta}^*(g,u) = \frac{d}{dt}\big(R^\vartheta_{\tilde{\chi}(\exp(t\eta))}(g,u)\big)|_{t=0}$) for all $g \in G$ and $u \in U$. Then $\hat{\alpha}$ (resp. $\tilde{\alpha}$) is called tensorial iff for all $h \in H$ and $\eta \in \mathfrak{h}$

$$R^{\chi *}_h \hat{\alpha} = \hat{\alpha} \quad \big(\text{resp.} \quad {R^\vartheta_{\tilde{\chi}(h)}}^* \tilde{\alpha} = \tilde{\alpha}\big), \tag{2.30}$$

$$\hat{\alpha}(\hat{\eta}^*) = 0 \quad \big(\text{resp.} \quad \tilde{\alpha}(\tilde{\eta}^*) = 0\big). \tag{2.31}$$

In particular, each pull-back $\kappa^*_\chi \alpha$ (resp. $\tilde{\kappa}^*_\chi \alpha$) of a $\mathfrak{u}$-valued 1-form α on P_χ is tensorial. It is well-known that the pull-back is a linear bijection of the vector space of all $\mathfrak{u}$-valued 1-forms on P_χ and the vector space of the tensorial forms on the total space of the H-bundle, see e.g. [21, p.76]. Concentrating on the case $G_\vartheta \times U$, we see that the affine space of all G-ϑ-equivariant connection 1-forms on the principal U-bundle P_χ is in bijection (vial pull-back with $\tilde{\kappa}_\chi$) with the affine space of all $\mathfrak{u}$-valued 1-forms $\tilde{\alpha}$ on the Lie group $G_\vartheta \times U$ satisfying the tensoriality conditions (2.30) and (2.31), and in addition the following conditions for which we use equations (1.4), (1.5), and (2.3) for a G-ϑ-equivariant connection), and the fact that $\tilde{\kappa}_\chi$ intertwines the left G-action and the right U-action on $G_\vartheta \times U$ and on P_χ, i.e. $\tilde{\kappa}_\chi \circ L^\vartheta_{(g_0, e_U)} = \ell'_{g_0} \circ \tilde{\kappa}_\chi$ and $\tilde{\kappa}_\chi \circ L^\vartheta_{(e, u_0^{-1})} = r_{u_0} \circ \tilde{\kappa}_\chi$ according to eqs (2.16) and (2.17) for all $g_0 \in G$, $u_0 \in U$, and $\zeta \in \mathfrak{u}$:

$${L^\vartheta_{(e,u_0^{-1})}}^* \tilde{\alpha} = Ad_{u_0^{-1}} \circ \tilde{\alpha}, \tag{2.32}$$

$$\tilde{\alpha}(\tilde{\zeta^*}) = \zeta, \tag{2.33}$$

$${L^\vartheta_{(g_0,e_U)}}^* \tilde{\alpha} = T_{e_U}\vartheta_{g_0} \circ \tilde{\alpha}. \tag{2.34}$$

where the fundamental field $\tilde{\zeta}^*$ is defined by $\tilde{\zeta}^*(g,u) = \frac{d}{dt}\big(L^{\vartheta}_{(e,\exp(-t\zeta)}(g,u)\big)|_{t=0}$. In the ensuing computations the following smooth map $\dot{\vartheta} : U \to \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u})$ appears very often: for all $u \in U$, $\xi \in \mathfrak{g}$) set

$$\dot{\vartheta}(u)(\xi) = \dot{\vartheta}_u(\xi) := \frac{d}{dt}\big(u^{-1}\vartheta_{exp(t\xi)}(u)\big)|_{t=0}. \tag{2.35}$$

which is well-defined because $t \mapsto u^{-1}\vartheta_{exp(t\xi)}(u)$ is a smooth curve emanating at the unit element $e_U \in U$. Note that $\dot{\vartheta}$ vanishes for the trivial case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$. It can be seen as the evaluation of the Maurer-Cartan form on U on the fundamental field of the left G-action on U.

There is the following

Proposition 2.3. *With the above definitions: Let $\chi \in \mathcal{P}_{\vartheta}(H,U)$, and P_χ the corresponding G-ϑ-equivariant U-bundle over G/H. Let $\tilde{\kappa}_\chi : G_\vartheta \times U \to P_\chi$ be the canonical projection. Then the following affine spaces are in bijection:*

1. *The affine space of all G-ϑ-equivariant connection 1-forms on P_χ.*
2. *The affine space of all linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ satisfying the following equations for all $h \in H$, $\xi \in \mathfrak{g}$, and $\eta \in \mathfrak{h}$:*

$$\begin{aligned} \varphi[\chi,\mathsf{p}(h)](\xi) &:= T_{e_U}\vartheta_h\Big(Ad_{\chi(h)}\big(\mathsf{p}(Ad_{h^{-1}}\xi)\big)\Big) - \mathsf{p}(\xi) + \dot{\vartheta}_{\chi(h^{-1})}(\xi) \\ &= 0, \end{aligned} \tag{2.36}$$

$$\mathsf{p}(\eta) - T_e\chi(\eta) = 0. \tag{2.37}$$

The bijection is given by

$$\alpha \mapsto \Big(\xi \mapsto \big(\tilde{\kappa}_\chi^*\alpha\big)_{(e,e_U)}(\xi,0)\Big).$$

and in the other direction by $\mathsf{p} \mapsto \alpha[\chi,\mathsf{p}]$ where its pull-back to the semidirect product $G_\vartheta \times U$ reads

$$\big(\tilde{\kappa}_\chi^*\alpha[\chi,\mathsf{p}]\big)_{(g,u)}\big(T_{(e,e_U)}L^{\vartheta}_{(g,u)}(\xi,\zeta)\big) = T_{e_U}\vartheta_g\Big(Ad_u\big(\mathsf{p}(\xi)-\zeta\big)\Big) \tag{2.38}$$

for all $g \in G$, $u \in U$, $\xi \in \mathfrak{g}$, and $\zeta \in \mathfrak{u}$; and its pull-back to $G \times U$ reads

$$\big(\kappa_\chi^*\alpha[\chi,\mathsf{p}]\big)_{(g,u)}\big(T_eL_g(\xi), T_{e_U}L_u(\zeta)\big) = T_{e_U}\vartheta_g\Big(Ad_{u^{-1}}\big(\mathsf{p}(\xi)\big) + \dot{\vartheta}_u(\xi) + \zeta\Big) \tag{2.39}$$

for all $g \in G$, $u \in U$, $\xi \in \mathfrak{g}$, and $\zeta \in \mathfrak{u}$.

Proof: It is easy to compute the following identities in the semidirect product for all $g \in G$, $u \in U$, $\xi \in \mathfrak{g}$, and $\zeta \in \mathfrak{u}$

$$T_{(e,e_U)}L^{\vartheta}_{(g,u)}(\xi,\zeta)\big) = \Big(T_eL_g(\xi), \;\; T_{e_U}L_u(\zeta) - T_{e_U}L_u\big(\dot{\vartheta}_u(\xi)\big)\Big), \tag{2.40}$$

$$(2.41)\qquad T_{(e,e_U)}R^{\vartheta}_{(g,u)}(\xi,\zeta)\big) \;=\; \Big(T_eR_g(\xi),\;\; T_{e_U}R_u\big(T_{e_U}\vartheta_{g^{-1}}(\zeta)\big)\Big),$$

$$(2.42)\qquad \tilde{\zeta}^*(g,u) \;=\; T_{(e,e_U)}L^{\vartheta}_{(g,u)}\Big(0,\;\; -Ad_{u^{-1}}\big(\vartheta_{g^{-1}}(\zeta)\big)\Big)$$

We deduce the following formula for the adjoint representation in the semidirect product which we shall use very often:

$$(2.43)\qquad Ad^{\vartheta}_{(g,u)}(\xi,\zeta) = \Big(Ad_g(\xi),\;\; T_{e_U}\vartheta_g\big(Ad_u(\zeta)\big) - T_{e_U}\vartheta_g\big(Ad_u(\dot{\vartheta}_u(\xi))\big)\Big).$$

Moreover, we get for all $\eta \in \mathfrak{h}$

$$(2.44)\qquad \tilde{\eta}^*(g,u) = T_{(e,e_U)}L^{\vartheta}_{(g,u)}\Big(\eta,\;\; T_e\chi(\eta)\Big).$$

Let first α be a G-ϑ-equivariant connection 1-form on P_χ, and set $\tilde{\alpha} = \tilde{\kappa}_\chi^*\alpha$. Since $(g,e_U)(e,u) = (g,u)$ for all $g \in G$ and $u \in U$ conditions (2.32) and (2.34) show that

$$L^{\vartheta}_{(g,u)}{}^*\tilde{\alpha} = L^{\vartheta}_{(e,u)}{}^*\big(L^{\vartheta}_{(g,e_U)}{}^*\tilde{\alpha}\big) = L^{\vartheta}_{(e,u)}{}^*\big(T_{e_U}\vartheta_g \circ \tilde{\alpha}\big) = T_{e_U}\vartheta_g \circ Ad_u \circ \tilde{\alpha},$$

hence writing the linear map $\tilde{\alpha}_{(e,e_U)} : \mathfrak{g}\times\mathfrak{u} \to \mathfrak{u}$ as (p,q) with linear maps $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ and $\mathsf{q} : \mathfrak{u} \to \mathfrak{u}$, we get

(2.45)

$$\tilde{\alpha}_{(g,u)}\big(T_{(e,e_U)}L^{\vartheta}_{(g,u)}(\xi,\zeta)\big) = T_{e_U}\vartheta_g\Big(Ad_u\Big(\mathsf{p}(\xi)+\mathsf{q}(\zeta)\Big)\Big) = T_{e_U}\vartheta_g\Big(Ad_u\Big(\mathsf{p}(\xi)-\zeta\Big)\Big)$$

because condition (2.33) reads for all $g \in G$, $u \in U$, $\zeta \in \mathfrak{u}$

$$\zeta = \tilde{\alpha}_{(g,u)}\big(\tilde{\zeta}^*(g,u)\big) \overset{(2.42)}{=} -\mathsf{q}(\zeta).$$

For future use, let us define for any linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ the linear map $\tilde{\mathsf{p}}$: $\mathfrak{g}\times\mathfrak{u} \to \mathfrak{u}$ by (for all $\xi \in \mathfrak{g}$ and $\zeta \in \mathfrak{u}$):

$$(2.46)\qquad \tilde{\mathsf{p}}(\xi,\zeta) := \mathsf{p}(\xi) - \zeta.$$

which at present is of course equal to $\tilde{\alpha}_{(e,e_U)}$.
Conversely, it is easy to see that for any choice of linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ the right hand side of equation (2.45) can be used as a definition of a $\mathfrak{u}$-valued 1-form $\tilde{\alpha}$ on the Lie group $G_\vartheta \times U \to P_\chi$ which automatically satisfies conditions (2.32), (2.34), and (2.42) since any Lie group is parallelizable.
Since left and right multiplications in any Lie group commute, the condition (2.30) $(R^{\vartheta}_{\tilde{\chi}(h)}{}^*\tilde{\alpha})_{(g,u)} = \tilde{\alpha}_{(g,u)}$ for any $g \in G$, $u \in U$ is in fact equivalent (thanks to the identities (2.32) and (2.34)) to $(R^{\vartheta}_{\tilde{\chi}(h)}{}^*\tilde{\alpha})_{(e,e_U)} = \tilde{\alpha}_{(e,e_U)}$. Now

$$\big(R^{\vartheta}_{\tilde{\chi}(h)}{}^*\tilde{\alpha}\big)_{(e,e_U)}(\xi,\zeta) - \tilde{\alpha}_{(e,e_U)}(\xi,\zeta)$$

$$\begin{aligned}
&= \quad \tilde{\alpha}_{\tilde{\chi}(h)}\Big(T_{(e,e_U)}L^{\vartheta}_{\tilde{\chi}(h)}\Big(Ad^{\vartheta}_{\tilde{\chi}(h^{-1})}\big(\xi,\zeta\big)\Big)\Big) - \tilde{\alpha}_{(e,e_U)}(\xi,\zeta)\\
&= \quad T_{e_U}\vartheta_h\Big(Ad_{\chi(h)}\Big(\tilde{\mathsf{p}}\Big(Ad^{\vartheta}_{\tilde{\chi}(h^{-1})}\big(\xi,\zeta\big)\Big)\Big)\Big) - \tilde{\mathsf{p}}(\xi,\zeta)\\
&\overset{(2.38),(2.43)}{=} \quad T_{e_U}\vartheta_h\Big(Ad_{\chi(h)}\Big(\mathsf{p}\big(Ad_{h^{-1}}(\xi)\big) - Ad_{\chi(h)^{-1}}\big(T_{e_U}\vartheta_{h^{-1}}\big(\zeta - \dot{\vartheta}_{\chi(h^{-1})}(\xi)\big)\big)\Big)\Big)\\
&\qquad\qquad - \mathsf{p}(\xi) + \zeta \quad = \quad \varphi[\chi,\mathsf{p}](h)\big(\xi\big)
\end{aligned}$$

using the identities $\vartheta_{h^{-1}}(\chi(h^{-1})) = \chi(h)^{-1}$ for any $h \in H$. This shows that invariance of $\tilde{\alpha}$ by H is equivalent to eqn (2.36) on p. Finally, for all $\eta \in \mathfrak{h}$ we get $\tilde{\alpha}_{(g,u)}(\tilde{\eta}^*(g,u)) = T_{e_U}\vartheta_g\big(Ad_u\big(\mathsf{p}(\eta) - T_e\chi(\eta)\big)\big)$ whence eqn (2.37) is equivalent to (2.31).
The computation of formula (2.39) is straight-forward using eqn (2.38) and the relation between κ_χ and $\tilde{\kappa}_\chi$, eqn (2.29). □

In the particular case of G-invariant connections (i.e. $\vartheta_g = \mathrm{id}_U$ for all $g \in G$) the above Proposition had already been formulated by H.-C. Wang in 1958, [36].

2.4.2 Smooth Lie group cohomology and Chevalley-Eilenberg cohomology

In order to understand the two conditions (2.36) and (2.37) on the linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ we recall the definition of *smooth Hochschild cohomology of the Lie group* H: let $\mathbf{V}$ be a smooth finite-dimensional H-module, i.e. a finite-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and a smooth Lie group homomorphism $H \to GL(\mathbf{V})$. Define the smooth Lie group (co)complex by setting $CG^k(H,\mathbf{V}) = \{0\}$ for each strictly negative integer k, and

$$CG^0(H,\mathbf{V}) := \mathbf{V} \ \text{ and } \ CG^k(H,\mathbf{V}) := \mathcal{C}^\infty(\underbrace{H \times \cdots \times H}_{k \text{ factors}},\mathbf{V}) \ \ \forall\, k \in \mathbb{N}\setminus\{0\},$$

and let $CG(H,\mathbf{V}) := \oplus_{k\in\mathbb{N}} CG^k(H,\mathbf{V})$. Recall the *coboundary operator* $\delta : CG(H,\mathbf{V}) \to CG(H,\mathbf{V})$ of degree 1 which is defined on each $f \in CG^k(H,\mathbf{V})$ by

$$\begin{aligned}
(\delta f)(h_1,\ldots,h_{k+1}) \;=\;& h_1\big(f(h_2,\ldots,h_{k+1})\big)\\
&+\sum_{r=1}^{k}(-1)^r f(h_1,\ldots,h_r h_{r+1},\ldots,h_{k+1})\\
&+(-1)^{k+1} f(h_1,\ldots,h_k).
\end{aligned} \tag{2.47}$$

It is easy to check that $\delta^2 = 0$, and define the *kth cohomology group* $HG^k(H,\mathbf{V})$ *of* H *with values in* $\mathbf{V}$ by

$$HG^k(H,\mathbf{V}) := \frac{\mathrm{Ker}\big(\delta : CG^k(H,\mathbf{V}) \to CG^{k+1}(H,\mathbf{V})\big)}{\mathrm{Im}\big(\delta : CG^{k-1}(H,\mathbf{V}) \to CG^k(H,\mathbf{V})\big)} =: \frac{ZG^k(H,\mathbf{V})}{BG^k(H,\mathbf{V})} \tag{2.48}$$

Recall that the elements of the 'numerator' of the above factor space are called *k-cocycles* and the elements of the 'denominator' are called *k-coboundaries.* In particular for $k = 0$ we have $HG^0(H, \mathbf{V}) \cong ZG^0(H, \mathbf{V})$ which is equal to the subspace of all *invariants* of $\mathbf{V}$, $\mathbf{V}^H$.
The above cohomology framework being the most naive one, there are other choices for the cochain space: a map $f : H^{\times k} \to V$ is called *locally smooth* iff there is an open neighbourhood of $(e, \dots, e) \in H^{\times k}$ (depending on f) such that the restriction of f to that neighbourhood is smooth. There are still others, see e.g. the article [34] for a good review of this.

Next recall for any Lie algebra $(\mathfrak{h}, [\ ,\ ])$ over any field $\mathbb{K}$ (which is not necessarily finite-dimensional), and any $\mathfrak{h}$-module V (where we write $v \mapsto \dot{\rho}_\eta(v)$ for the module map) its *Chevalley-Eilenberg complex* by setting $C^k_{CE}(\mathfrak{h}, V) = \{0\}$ for each strictly negative integer k, and

$$C^0_{CE}(\mathfrak{h}, \mathbf{V}) := \mathbf{V} \ \text{ and } \ C^k_{CE}(\mathfrak{h}, \mathbf{V}) := \mathbf{Hom}_{\mathbb{K}}(\Lambda^k \mathfrak{h}, \mathbf{V}) \ \ \forall\, k \in \mathbb{N} \setminus \{0\},$$

and let $C_{CE}(\mathfrak{h}, \mathbf{V}) := \oplus_{k \in \mathbb{N}} C^k_{CE}(\mathfrak{h}, \mathbf{V})$. Recall the *coboundary operator* $\delta_{CE} : C_{CE}(\mathfrak{h}, \mathbf{V}) \to C_{CE}(\mathfrak{h}, \mathbf{V})$ of degree 1 which is defined on each $\varphi \in CG^k(H, \mathbf{V})$ by

$$(\delta_{CE}\varphi)(\eta_1 \wedge \cdots \wedge \eta_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \dot{\rho}_{\eta_i}\big(\varphi(\eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_{k+1}\big) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \varphi\big([\eta_i, \eta_j] \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \widehat{\eta_j} \wedge \cdots \wedge \eta_{k+1}\big) \tag{2.49}$$

It is easy to check that $\delta^2_{CE} = 0$, and define the *kth cohomology group* $H^k_{CE}(\mathfrak{h}, \mathbf{V})$ *of* $\mathfrak{h}$ *with values in* $\mathbf{V}$ by

$$H^k_{CE}(\mathfrak{h}, \mathbf{V}) := \frac{\mathrm{Ker}\big(\delta_{CE} : C^k_{CE}(\mathfrak{h}, \mathbf{V}) \to C^{k+1}_{CE}(\mathfrak{h}, \mathbf{V})\big)}{\mathrm{Im}\big(\delta_{CE} : C^{k-1}_{CE}(\mathfrak{h}, \mathbf{V}) \to C^k_{CE}(\mathfrak{h}, \mathbf{V})\big)} =: \frac{Z^k_{CE}(\mathfrak{h}, \mathbf{V})}{B^k_{CE}(\mathfrak{h}, \mathbf{V})} \tag{2.50}$$

Recall that the elements of the 'numerator' of the above factor space are called *k-cocycles* and the elements of the 'denominator' are called *k-coboundaries.* In particular for $k = 0$ we have $H^0_{CE}(\mathfrak{h}, \mathbf{V}) \cong Z^0_{CE}(\mathfrak{h}, \mathbf{V})$ which is equal to the subspace of all *invariants* of $\mathbf{V}$, $\mathbf{V}^{\mathfrak{h}}$.

Note also that there is the following chain map $\mathbf{D}$ to the Chevalley-Eilenberg cohomology complex of the Lie algebra $\mathfrak{h}$ with values in the $\mathfrak{h}$-module $\mathbf{V}$: let $f \in CG^k(H, \mathbf{V})$, $\eta \in \mathfrak{h}$, and $i \in \mathbb{N}$ such that $1 \le i \le k$, then let $\eta^{(i)}$ denote the left invariant vector field on the Lie group $H^{\times k}$ whose value at $(e, \dots, e)$ is given by $(0, \dots, 0, \eta, 0, \dots, 0)$ where η appears in the ith factor. Define

$$\big(\mathbf{D}_k(f)\big)(\eta_1 \wedge \cdots \wedge \eta_k) := \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\big(\eta^{(1)}_{\sigma(1)}\big(\eta^{(2)}_{\sigma(2)}\big(\cdots\big(\eta^{(k)}_{\sigma(k)}(f)\big)\cdots\big)(e, \dots, e). \tag{2.51}$$

It is not hard to check that this is a chain map from the (locally) smooth cochains to the Chevalley-Eilenberg cochains.

2.4.3 Atiyah classes as obstructions to the existence of G-ϑ equivariant connections

Now note that the map $\Psi^\chi : H \to \mathrm{GL}(\mathfrak{u})$ defined by

$$\Psi^\chi_h = T_{e_U}\vartheta_h \circ Ad_{\chi(h)} \tag{2.52}$$

is a smooth representation because $\big(0, \Psi^\chi_h(\zeta)\big) = Ad^\vartheta_{\tilde{\chi}(h)}(0,\zeta)$, see eqn (2.43). Moreover, there is the exact sequence of H-modules (w.r.t. the adjoint representation of H on $\mathfrak{g}$, $\mathfrak{h}$, and hence on $\mathfrak{g}/\mathfrak{h}$)

$$\{0\} \to \mathfrak{h} \stackrel{\iota}{\to} \mathfrak{g} \stackrel{\varpi}{\to} \mathfrak{g}/\mathfrak{h} \to \{0\},$$

and since they are vector space there results the exact sequence of H-modules

$$\{0\} \to \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h},\mathfrak{u}) \stackrel{\varpi^*}{\to} \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u}) \stackrel{\iota^*}{\to} \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h},\mathfrak{u}) \to \{0\},$$

where H acts on the target modules of the above **Hom**-spaces by means of Ψ^χ. From this sequence we get a short exact sequence of Hochschild complexes

$$\begin{aligned}\{0\} \to CG\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h},\mathfrak{u})\big) &\stackrel{\widehat{\varpi^*}}{\to} CG\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u})\big) \\ &\stackrel{\widehat{\iota^*}}{\to} CG\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h},\mathfrak{u})\big) \to \{0\}\end{aligned} \tag{2.53}$$

where the Hochschild coboundary δ depends on χ, and we shall sometimes write δ^χ. Recall the following important Hochschild cochains:

$$T_e\chi \in CG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h},\mathfrak{u})\big) \tag{2.54}$$

$$\dot{\vartheta}_{\chi((\cdot)^{-1})} : h \mapsto \big(\xi \mapsto \dot{\vartheta}_{\chi(h^{-1})}(\xi)\big) \in CG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u})\big) \tag{2.55}$$

Lemma 2.4. *With the above notations:*

$$\delta\big(\dot{\vartheta}_{\chi((\cdot)^{-1})}\big) = 0 \tag{2.56}$$

$$\delta\big(T_e\chi\big) + \widehat{\iota^*}\big(\dot{\vartheta}_{\chi((\cdot)^{-1})}\big) = 0 \tag{2.57}$$

Proof: The first equation follows from the identity

$$Ad^\vartheta_{\tilde{\chi}(h_2^{-1}h_1^{-1})}(\xi,0) = \Big(Ad^\vartheta_{\tilde{\chi}(h_2^{-1})} \circ Ad^\vartheta_{\tilde{\chi}(h_1^{-1})}\Big)(\xi,0)$$

for all $h_1, h_2 \in H$, $\xi \in \mathfrak{g}$, see eqn (2.43). The second equation is deduced from the fact that $\tilde{\chi}$ is a homomorphism of Lie groups whence its derivative $T_e\tilde{\chi}$ intertwines the adjoint actions, viz.

$$T_e\tilde{\chi} \circ Ad_h = Ad^\vartheta_{\tilde{\chi}(h)} \circ T_e\tilde{\chi}$$

for all $h \in H$: it suffices to look at the $\mathfrak{u}$-component. □

There is now a characteristic class $c_{G,H,U,\vartheta,\chi}$ in $HG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$ defined as follows: since ι^* is surjective, we can choose a linear map $\mathsf{p} \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{u})$ with $\iota^*\mathsf{p} = T_e\chi$. We recall for all $h \in H$

$$\varphi[\chi, \mathsf{p}](h) = \Psi_h^\chi \circ \tilde{\mathsf{p}} \circ Ad^\vartheta_{\tilde{\chi}(h^{-1})} - \tilde{\mathsf{p}} = \delta(\mathsf{p})(h) + \dot{\vartheta}_{\chi(h^{-1})} \tag{2.58}$$

and get

$$\widehat{\iota^*}\big(\varphi[\chi, \mathsf{p}]\big) = \delta\big(\widehat{\iota^*}(\mathsf{p})\big) + \widehat{\iota^*}\big(\dot{\vartheta}_{\chi((\cdot)^{-1})}\big) = \delta\big(T_e\chi\big) + \widehat{\iota^*}\big(\dot{\vartheta}_{\chi((\cdot)^{-1})}\big) \overset{(2.57)}{=} 0$$

hence –using the fact that the sequence (2.53) is exact– there is a unique $f \in CG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$ such that

$$\widehat{\varpi^*}(f) = \varphi[\chi, \mathsf{p}] = \delta(\mathsf{p}) + \dot{\vartheta}_{\chi((\cdot)^{-1})}. \tag{2.59}$$

Thanks to (2.56) we get $0 = \delta\big(\widehat{\varpi^*}(f)\big) = \widehat{\varpi^*}\big(\delta(f)\big)$ whence $\delta(f) = 0$ since $\widehat{\varpi^*}$ is injective. For another choice p' in $\mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{u})$, note that $\iota^*(\mathsf{p}' - \mathsf{p}) = 0$ hence –again by exactness of the Hochschild complex, eqn (2.53)– there is a unique $g \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})$ with $\varpi^*(g) = \mathsf{p}' - \mathsf{p}$. It follows that

$$\widehat{\varpi^*}(f' - f)) = \delta\big(\mathsf{p}' - \mathsf{p}\big) = \widehat{\varpi^*}\big(\delta(g)\big) \tag{2.60}$$

whence $f' - f = \delta(g)$ since $\widehat{\varpi^*}$ is injective. It follows that the cohomology class $[f]$ of the 1-cocycle f does not depend on the choice of p, and hence the definition

$$c_\chi := c_{G,H,U,\vartheta,\chi} := [f] \in HG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big) \tag{2.61}$$

makes sense and is called the *Atiyah class* (of $(G, H, U, \vartheta, \chi)$).
Note that in the important particular case $\vartheta_g = \mathrm{id}_U$ for all $g \in G$ of an *invariant connection*, the map χ is already a homomorphism of Lie groups, whence $T_e\chi$ intertwines the adjoint actions, so it is a 0-cocycle and defines a cohomology class $[T_e\chi]$ in $HG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathfrak{u})\big)$. The image of this class under the connecting homomorphism

$$HG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathfrak{u})\big) \to HG^1\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$$

in the long exact cohomology sequence coincides with the Atiyah class.

The relation of the Atiyah class with the characterization of G-ϑ-equivariant connections in Proposition 2.3 is contained in the following

Proposition 2.5. *With the hypotheses of Proposition 2.3:*

1. *There is a G-ϑ-equivariant connection in the bundle P_χ if and only if the Atiyah class $c_{G,H,U,\vartheta,\chi}$ vanishes.*

2. *In case the Atiyah class vanishes: the tangent space of the affine space of all G-ϑ-equivariant connections in the bundle P_χ is isomorphic to*

$$HG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big) \cong ZG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$$

Proof: **1.** According to Proposition 2.3 a G-ϑ-equivariant connection 1-form on P_χ exists if and only if the linear map $\mathsf{p} \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{u})$ satisfies the two conditions (2.36) and (2.37) which in fact can be expressed in cohomological terms as

$$\delta(\mathsf{p}) + \dot{\vartheta}_{\chi((\cdot)^{-1})} = 0, \tag{2.62}$$

$$\iota^*\mathsf{p} = T_e\chi. \tag{2.63}$$

Now if such a map p satisfying the preceding conditions exists, it is clear that the map f in eqn (2.59) vanishes, hence its class, the Atiyah class, vanishes. Conversely, suppose that the Atiyah class $[f]$ vanishes. According to the definition of this class it follows that there is a map $\mathsf{p}' \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{u})$ such that $\iota^*\mathsf{p}' = T_e\chi$ and a map $g \in \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})$ such that

$$\widehat{\varpi^*}(f) = \widehat{\varpi^*}\big(\delta(g)\big) = \delta\big(\widehat{\varpi^*}(g)\big) = \delta(\mathsf{p}') + \dot{\vartheta}_{\chi((\cdot)^{-1})}.$$

It is immediate that the linear map $\mathsf{p} := \mathsf{p}' - \widehat{\varpi^*}(g)$ satisfies the two above conditions (2.62) and (2.63) whence a G-ϑ-equivariant connection 1-form exists on P_χ.
2. Suppose that p and p' satisfy (2.62) and (2.63). Then their difference $\mathsf{p}' - \mathsf{p}$ is a cococycle satisfying $\iota^*(\mathsf{p}' - \mathsf{p}) = 0$ whence there is a cocycle $g \in ZG^0\big(H, \mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$ such that $\varpi^* g = \mathsf{p}' - \mathsf{p}$. Conversely it is clear that for any such cocycle g the sum $\mathsf{p} + \varpi^* g$ satisfies (2.62) and (2.63) if p does. □

2.4.4 Lie algebra versions

One gets a '*Lie algebra version*' or an *infinitesimal version* of the preceding Atiyah classes by replacing group elements by the exponential functions of Lie algebra elements times a parameter t and differentiating w.r.t. t at $t = 0$; more precisely, fix the following data: let $(\mathfrak{g}, [\ ,\ ])$ and $(\mathfrak{u}, [\ ,\ ])$ be a Lie algebras (not necessarily finite-dimensional), let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. In order to get an analogue of the automorphic G-action ϑ (2.1) define the linear map $\ddot{\vartheta} : \mathfrak{g} \to \mathbf{Hom}_{\mathbb{K}}(\mathfrak{u}, \mathfrak{u})$ as the following 'second derivative of ϑ' for all $\xi \in \mathfrak{g}$ and $\zeta \in \mathfrak{u}$:

$$\begin{aligned}\ddot{\vartheta}_\xi(\zeta) &:= \frac{\partial}{\partial s}\Big(T_{e_U}\vartheta_{\exp(s\xi)}(\zeta)\Big)\Big|_{s=0} = \frac{\partial^2}{\partial s \partial t}\Big(\vartheta_{\exp(s\xi)}\big(\exp(t\zeta)\big)\Big)\Big|_{s=0=t} \\ &= \frac{\partial}{\partial t}\Big(\dot{\vartheta}_{\exp(t\zeta)}(\xi)\Big|_{t=0}.\end{aligned}$$

It follows immediately that $\ddot{\vartheta}$ is a *derivational Lie algebra representation* in the following sense: for all $\xi \in \mathfrak{g}$ and $\zeta \in \mathfrak{u}$, satisfying (for all $\xi, \xi' \in \mathfrak{g}$ and $\zeta, \zeta' \in \mathfrak{u}$)

$$\ddot{\vartheta}_{[\xi,\xi']}(\zeta) = \ddot{\vartheta}_\xi\big(\ddot{\vartheta}_{\xi'}(\zeta)\big) - \ddot{\vartheta}_{\xi'}\big(\ddot{\vartheta}_\xi(\zeta)\big) \quad \text{and} \quad \ddot{\vartheta}_\xi\big([\zeta, \zeta']\big) = [\ddot{\vartheta}_\xi(\zeta), \zeta'] + [\zeta, \ddot{\vartheta}_\xi(\zeta')]. \tag{2.64}$$

In our more general situation of Lie algebras we fix such a derivational Lie algebra representation $\ddot{\vartheta}$. Note the important particular case $\ddot{\vartheta} = 0$ corresponding to trivial ϑ. With these data one can form the *semidirect sum* $\mathfrak{g} \oplus \mathfrak{u}$ of the two Lie algebras: on the direct sum of the vector spaces the bracket is defined as follows for all $\xi, \xi' \in \mathfrak{g}$ and $\zeta, \zeta' \in \mathfrak{u}$:

$$[(\xi,\zeta),(\xi',\zeta')] := \big([\xi,\xi'], \ddot{\vartheta}_\xi(\zeta') - \ddot{\vartheta}_{\xi'}(\zeta) + [\zeta,\zeta']\big), \tag{2.65}$$

and it is easy to check that this is a Lie bracket. Next, in order to get an analogue of the crossed homomorphism $\chi : H \to U$ we consider first its derivative $\dot{\chi} = T_e\chi$ at $e \in H$. Since $\tilde{\chi} : H \to G_\vartheta \times U$ defined by $\tilde{\chi}(h) = \big(h, \chi(h)\big)$ is a homomorphism of Lie groups, it follows that its derivative is a morphism of Lie algebras, hence upon using the above semidirect sum structure (2.65) we get the following identity of a *crossed Lie algebra homomorphism w.r.t.* $\ddot{\vartheta}$, i.e. for all $\eta, \eta' \in \mathfrak{h}$:

$$[\dot{\chi}(\eta), \dot{\chi}(\eta')] - \dot{\chi}([\eta,\eta']) + \ddot{\vartheta}_\eta\big(\dot{\chi}(\eta')\big) - \ddot{\vartheta}_{\eta'}\big(\dot{\chi}(\eta)\big) = 0. \tag{2.66}$$

Again in our more general situation we fix a crossed Lie algebra homomorphism $\dot{\chi}$. Having fixed the data $(\mathfrak{g}, \mathfrak{h}, \mathfrak{u}, \ddot{\vartheta}, \dot{\chi})$ we get the following: the linear map $\psi^{\dot{\chi}} : \mathfrak{h} \to \mathbf{Hom}_{\mathbb{K}}(\mathfrak{u},\mathfrak{u})$, written $\psi^{\dot{\chi}}_\eta(\zeta)$ for all $\eta \in \mathfrak{h}$ and $\zeta \in \mathfrak{u}$ and defined by

$$\psi^{\dot{\chi}}_\eta(\zeta) = \ddot{\vartheta}_\eta(\zeta) + ad_{\dot{\chi}(\eta)}(\zeta), \tag{2.67}$$

is readily checked to be a representation of $\mathfrak{h}$ in $\mathfrak{u}$. The formula can be obtained by differentiating (2.52). Hence we have the Chevalley-Eilenberg complexes of $\mathfrak{h}$ with values in $\mathbf{Hom}_{\mathbb{K}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})$, in $\mathbf{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{u})$, and in $\mathbf{Hom}_{\mathbb{K}}(\mathfrak{h}, \mathfrak{u})$. The linear map $\ddot{\vartheta} \circ (\mathrm{id}_\mathfrak{g} \otimes \dot{\chi})$ is readily checked to be Chevalley-Eilenberg 1-cocycle with values in $\mathbf{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{u})$, i.e.

$$\delta_{CE}\big(\ddot{\vartheta} \circ (\mathrm{id}_\mathfrak{g} \otimes \dot{\chi})\big) = 0, \tag{2.68}$$

(analogue of the group cocycle $h \mapsto \big(\xi \mapsto \dot{\vartheta}_{\chi(h^{-1})}(\xi)\big)$, see equation (2.56). Moreover, as a direct consequence of eqn (2.66) we get the analogue of eqn (2.57),

$$\delta_{CE}(\dot{\chi}) - \widehat{\iota^*}\big(\ddot{\vartheta} \circ (\mathrm{id}_\mathfrak{g} \otimes \dot{\chi})\big). = 0. \tag{2.69}$$

Finally, given a linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ such that the restriction of p to $\mathfrak{h}$ is equal to $\dot{\chi}$ the Lie analogue of the map $\varphi[\chi, \mathsf{p}] : \mathfrak{g} \to \mathfrak{u}$ (see eqn (2.36) is given by

$$\varphi[\dot{\chi}, \mathsf{p}] = \delta_{CE}(\mathsf{p}) - \ddot{\vartheta} \circ (\mathrm{id}_\mathfrak{g} \otimes \dot{\chi}). \tag{2.70}$$

With these identities, a linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{h}$ may be called an *infinitesimal* $\mathfrak{g}$-$\ddot{\vartheta}$*-equivariant connection* iff the restriction of p to $\mathfrak{h}$ equals $\dot{\chi}$ and iff $\varphi[\dot{\chi}, \mathsf{p}] = 0$. Moreover, the whole homological reasoning of the preceding Subsubsection can be copied to define an Atiyah class $c_{\mathfrak{g},\mathfrak{h},\mathfrak{u},\ddot{\vartheta},\dot{\chi}}$ as an element of $H^1_{CE}\big(\mathfrak{h}, \mathbf{Hom}_{\mathbb{K}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})\big)$ which is the obstruction to the existence of an *infinitesimal* $\mathfrak{g}$-$\ddot{\vartheta}$*-equivariant connection.*

2.4.5 Equivalence of the categories $\mathcal{PC}_\vartheta(H,U)$ and $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$

We shall return to the Lie group case: before we can define the relevant categories, we have to see how the group U acts on the structures defined before: clearly, the vector space $\mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}\times\mathfrak{u},\mathfrak{u})$ is a left U-module by the obvious conjugation $\mathsf{P}\mapsto\hat{u}.\mathsf{P}=Ad_{\hat{u}}\circ\mathsf{P}\circ Ad^\vartheta_{(e,\hat{u}^{-1})}$ for all $\hat{u}\in U$. The image of the map $\mathsf{p}\mapsto\tilde{\mathsf{p}}$, see eqn (2.46) is an affine subspace of the vector space $\mathbf{Hom}_{\mathbb{R}}(\mathfrak{g}\times\mathfrak{u},\mathfrak{u})$, and formula (2.43) shows that it is invariant under the above U-action. Hence there is an affine U-action on the vector space $\mathbf{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{u})$, given by

$$\hat{u}.\mathsf{p}=Ad_{\hat{u}}\circ\mathsf{p}+\dot{\vartheta}_{\hat{u}^{-1}}.\overset{(2.43)}{=}Ad_{\hat{u}}\circ\big(\mathsf{p}-\dot{\vartheta}_{\hat{u}}\big). \tag{2.71}$$

Next, for $\hat{u}\in U$ consider the crossed homomorphism $u.\chi$. Then using $\widetilde{\hat{u}.\chi}(h)=(e,\hat{u})\tilde{\chi}(h)(e,\hat{u}^{-1})$ we first get for all $\hat{u}\in U$ that

$$\Psi_h^{\hat{u}.\chi}(\zeta)=\mathrm{pr}_2\circ Ad^\vartheta_{(e,\hat{u})\tilde{\chi}(h)(e,\hat{u}^{-1})}(0,\zeta)=\Big(Ad_{\hat{u}}\circ\Psi_h^\chi\circ Ad_{\hat{u}^{-1}}\Big)(\zeta),$$

and for all $\eta\in\mathfrak{h}$:

$$T_e(\hat{u}.\chi)(\eta)=\mathrm{pr}_2\big(Ad^\vartheta_{(e,\hat{u})}\big(T_e\tilde{\chi}(\eta)\big)\big)=\mathrm{pr}_2\big(Ad^\vartheta_{(e,\hat{u})}\big(\eta,T_e\chi(\eta)\big)\big)\overset{(2.43)}{=}Ad_{\hat{u}}\big(T_e\chi(\eta)-\dot{\vartheta}_{\hat{u}}(\eta)\big).$$

whence a comparison with equation (2.71) shows that for all $\hat{u}\in U$

$$T_e(\hat{u}.\chi)=\hat{u}.T_e\chi \tag{2.72}$$

where all the preceding considerations and definitions also work for Hom-spaces with $\mathfrak{g}$ replaced by $\mathfrak{h}$. An immediate consequence is the equation

$$\iota^*\mathsf{p}=T_e\chi\iff\iota^*\big(\hat{u}.\mathsf{p}\big)=T_e(\hat{u}.\chi).$$

Next, recall that the defining 1-cochain for the Atiyah class, $\varphi[\chi,\mathsf{p}]$, see eqn (2.58), can be written as

$$\varphi[\chi,\mathsf{p}](h)\big(\xi\big)=\Big(\Psi_h^\chi\circ\tilde{\mathsf{p}}\circ Ad^\vartheta_{\tilde{\chi}(h^{-1})}\Big)\big(\xi,\zeta\big)-\tilde{\mathsf{p}}(\xi,\zeta)$$

and is independent of $\zeta\in\mathfrak{u}$, see the proof of Proposition 2.3. We compute that for all $\hat{u}\in U$, $h\in H$, and $\xi\in\mathfrak{g}$:

$$\varphi[\hat{u}.\chi,\hat{u}.\mathsf{p}](h)\big(\xi\big)=Ad_{\hat{u}}\Big(\varphi[\chi,\mathsf{p}](h)\big(\xi\big)\Big). \tag{2.73}$$

Now observe that for each $\hat{u}\in U$ the map $f\mapsto Ad_{\hat{u}}\circ f$ from $\mathbf{Hom}_{\mathbb{K}}(W,\mathfrak{u})$ (for $W=\mathfrak{g}$, $\mathfrak{h}$, or $\mathfrak{g}/\mathfrak{h}$) intertwines the H-actions Ψ^χ and $\Psi^{\hat{u}.\chi}$. Therefore it induces a canoncial map $Ad'_{\hat{u}}$ on Hochschild cohomology. The above equation (2.73) implies that

$$c_{G,H,U,\vartheta,\hat{u}.\chi}=Ad'_{\hat{u}}\big(c_{G,H,U,\vartheta,\chi}\big). \tag{2.74}$$

We shall define two small categories: let $\mathcal{P}^0_\vartheta(H,U)$ be the following set:

$$\mathcal{P}^0 = \mathcal{P}^0_\vartheta(H,U) := \{\chi \in \mathcal{P}_\vartheta(H,U) \mid c_{G,H,U,\vartheta,\chi} = 0\}. \tag{2.75}$$

and for any two $\chi, \chi' \in \mathcal{P}^0_\vartheta(H,U)$

$$\mathbf{Hom}_{\mathcal{P}^0}(\chi,\chi') := \mathbf{Hom}_{\mathcal{P}}(\chi,\chi'). \tag{2.76}$$

which is well-defined thanks to eqn (2.74). Note also that $\mathcal{P}^0_\vartheta(H,U)$ is not empty because for the constant map (i.e. $\chi(h) = e_U$ for all $h \in H$)–which gives a trivial bundle– it follows that $T_e\chi = 0$ and $\dot{\vartheta}_{\chi(\cdot)^{-1}} = \dot{\vartheta}_{e_U} = 0$ whence $c_{G,H,U,\vartheta,\chi} = 0$. Hence $\mathcal{P}^0_\vartheta(H,U)$ is a full subcategory of $\mathcal{P}_\vartheta(H,U)$.
We now define a second small category by declaring its set of objects by

$$\begin{aligned}\mathcal{PC}_\vartheta(H,U) = \mathcal{PC} \;\; := \;\; & \{(\chi,\mathsf{p}) \in \mathcal{P}^0_\vartheta(H,U) \times \mathbf{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{u}) \mid \text{such that} \\ & \forall\, \eta \in \mathfrak{h} : \mathsf{p}(\eta) = T_e\chi(\eta) \text{ and } \varphi[\chi,\mathsf{p}] = 0\},\end{aligned} \tag{2.77}$$

and its set of morphisms by

$$\begin{aligned}&\mathbf{Hom}_{\mathcal{PC}}\big((\chi,\mathsf{p}),(\chi',\mathsf{p}')\big) := \\ &\qquad \{\hat{u} \in \mathbf{Hom}_{\mathcal{P}^0}(\chi,\chi') \mid \mathsf{p}' = \hat{u}.\mathsf{p} = Ad_{\hat{u}} \circ \mathsf{p} + \dot{\vartheta}_{\hat{u}^{-1}}\},\end{aligned} \tag{2.78}$$

where composition of morphism is given by group multiplication in U. It is clear from the above considerations that $\mathcal{PC}_\vartheta(H,U)$ is a category, and that the projection $(\chi,\mathsf{p}) \to \chi$ is a full covariant functor to $\mathcal{P}^0_\vartheta(H,U)$.

We wish to define covariant functors between the categories $\mathcal{PC}_\vartheta(H,U)$ and $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$: let $\mathbf{PC} : \mathcal{PC}_\vartheta(H,U) \to G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$ be the assignment

$$\mathbf{PC}(\chi,\mathsf{p}) = \big(P_\chi, \alpha[\chi,\mathsf{p}]\big) \quad \text{and} \quad \mathbf{PC}(\hat{u}) = P_{\hat{u}} \tag{2.79}$$

where $(\chi,\mathsf{p}),(\chi',\mathsf{p}') \in \mathcal{PC}_\vartheta(H,U)$, and $\hat{u} \in \mathbf{Hom}_{\mathcal{PC}}\big((\chi,\mathsf{p}),(\chi',\mathsf{p}')\big)$, see eqs (2.27) and (2.38). According to Proposition 2.3 the assignment $\mathbf{PC}$ is well-defined on the object level, and $\mathbf{PC}_{\hat{u}} = P_{\hat{u}} : P_\chi \to P_{\chi'} = P_{\hat{u}.\chi}$ is a morphism of G-ϑ-equivariant principal U-bundles over G/H according to Proposition 2.2. In order to show that it is a morphism in $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$, note first that eqs (2.15), (2.18), (2.27), and (2.29) imply that

$$P_{\hat{u}} \circ \tilde{\kappa}_\chi = \tilde{\kappa}_{\hat{u}.\chi} \circ R^\vartheta_{(e,\hat{u}^{-1})}$$

and for all $g \in G$, $u \in U$, $\xi \in \mathfrak{g}$, $\zeta \in \mathfrak{u}$:

$$\begin{aligned}&\Big(\tilde{\kappa}^*_\chi\big(P^*_{\hat{u}}(\alpha[\hat{u}.\chi,\hat{u}.\mathsf{p}])\big)\Big)_{(g,u)}\Big(T_{(e,e_U)}L^\vartheta_{(g,u)}(\xi,\zeta)\Big) \\ &= \quad \Big({R^\vartheta_{(e,\hat{u}^{-1})}}^*\big(\tilde{\kappa}^*_{\hat{u}.\chi}(\alpha[\hat{u}.\chi,\hat{u}.\mathsf{p}])\big)\Big)_{(g,u)}\Big(T_{(e,e_U)}L^\vartheta_{(g,u)}(\xi,\zeta)\Big) \\ &= \quad \big(\tilde{\kappa}^*_{\hat{u}.\chi}(\alpha[\hat{u}.\chi,\hat{u}.\mathsf{p}])\big)_{(g,u\hat{u}^{-1})}\Big(T_{(e,e_U)}L^\vartheta_{(g,u\hat{u}^{-1})}\big(Ad^\vartheta_{(e,\hat{u})}(\xi,\zeta)\big)\Big)\end{aligned}$$

$$\begin{aligned}
&\overset{(2.38),(2.43)}{=} && T_{e_U}\vartheta_g\Big(Ad_{u\hat{u}^{-1}}\Big((\hat{u}.\mathsf{p})(\xi) - Ad_{\hat{u}}(\zeta) + Ad_{\hat{u}}\big(\dot{\vartheta}_{\hat{u}}(\xi)\big)\Big)\Big)\\
&\overset{(2.71)}{=} && T_{e_U}\vartheta_g\Big(Ad_u\Big(\mathsf{p}(\xi) - \zeta\Big)\Big)\\
&= && \tilde{\kappa}^*_\chi\big(\alpha[\chi,\mathsf{p}]\big)_{(g,u)}\Big(T_{(e,e_U)}L^\vartheta_{(g,u)}(\xi,\zeta)\Big),
\end{aligned}$$

whence

$$P^*_{\hat{u}}(\alpha[\hat{u}.\chi,\hat{u}.\mathsf{p}]) = \alpha[\chi,\mathsf{p}], \tag{2.80}$$

showing that $P_{\hat{u}}$ preserves the connections and is thus a morphism in the category $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$. It is easy to check that **PC** is a covariant functor.
In order to define a covariant functor in the other direction $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}\to\mathcal{PC}_\vartheta(H,U)$ recall the functor **X** from $G\cdot\mathcal{PB}(U;\vartheta)_{G/H}\to\mathcal{P}_\vartheta(H,U)$, see Proposition 2.2, sending a G-ϑ-equivariant principal U-bundle over G/H, $(P,\tau,G/H,U,\vartheta)$, to the crossed homomorphism $\chi_P : H\to U$ upon choosing an element $y_P\in P_o$. Recall furthermore the natural isomorphism $\Phi_P : P_{\chi_P}\to P$ between the functors $\mathbf{P}\circ\mathbf{X}$ and $\mathrm{id}_{G\cdot\mathcal{PB}(U;\vartheta)_{G/H}}$, see eqn (2.28) in the proof of Proposition 2.2. Define the functor **XC** from $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}\to\mathcal{PC}_\vartheta(H,U)$ by

$$\mathbf{XC}(P,\tau,G/H,U,\vartheta,\alpha) = \Big(\chi_P,\quad \xi\mapsto\tilde{\kappa}^*_{\chi_P}\big(\Phi^*_P\alpha\big)_{(e,e_U)}\big(\xi,0\big)\Big) \tag{2.81}$$

Note that for any morphism Φ of G-ϑ-equivariant principal U-bundles over G/H the pull-back of a G-ϑ-equivariant connection 1-form with respect to Φ is again a G-ϑ-equivariant connection 1-form whence the image object of **XC** is an object in $\mathcal{PC}_\vartheta(H,U)$ thanks to Proposition 2.3. Define **XC** on morphisms Φ as the functor **X**, i.e. $\mathbf{XC}(\Phi)=\tilde{u}_\Phi\in U$, see the considerations preceding Proposition 2.2. A computation similar to the one leading to eqn (2.80) shows that $\tilde{u}_\Phi$ is a morphism in $\mathcal{PC}_\vartheta(H,U)$, i.e. maps p to $\tilde{u}_\Phi.\mathsf{p}$. It follows that **XC** is a covariant functor. We get the following analogue of Proposition 2.2:

Proposition 2.6. *The two functors* $\mathbf{PC} : (\chi,\mathsf{p})\to\big(P_\chi,\alpha[\chi,\mathsf{p}]\big)$ *and* **XC** *as defined above, see eqn (2.81), constitute an equivalence of the small category* $\mathcal{PC}_\vartheta(H,U)$ *and the large category* $G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}$*,*

$$\mathcal{PC}_\vartheta(H,U)\ \simeq\ G\cdot\mathcal{PBC}(U;\vartheta)_{G/H}.$$

Proof: Analogous to the proof of Proposition 2.2. □

2.4.6 Covariant derivatives in associated vector bundles for G-invariant connections

In this Subsubsection we shall –for simplicity– consider the situation where ϑ is trivial: let G,H,U as before, and $\chi : H\to U$ is a homomorphism of Lie groups.

Furthermore, suppose that the associated Atiyah class $c_{G,H,U,\chi}$ vanishes, and let $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ an H-invariant map, i.e. for all $h \in H$: $Ad_{\chi(h)} \circ \mathsf{p} = \mathsf{p} \circ Ad_h$. Suppose that there is a representation $\rho : U \to GL(V)$ of the Lie group U in a finite-dimensional vector space V. As before we shall write $\dot{\rho} : \mathfrak{u} \to \mathfrak{gl}(V)$ for the induced representation of the Lie algebra $\mathfrak{u}$ of U. Consider the G-equivariant principal U bundle $G_H[U]$ over G/H, and let E' denote the associated vector bundle $(G_H[U])_U[V]$ over G/H where U acts on V via ρ. Recall the morphism of principal bundles over G/H defined by $\Phi : G \to G_H[U] : g \mapsto [g, e_U]$, see eqn (1.2). By eqn (1.3) E' is naturally isomorphic as a G-equivariant vector bundle over G/H to the associated vector bundle $E = G_H[V]$ where H acts on V via the representation $\rho \circ \chi$ by means of the maps $\Phi_V : E \to E'$. Let ∇' denote the covariant derivative in the associated vector bundle E', see eqn (1.7). Let $\ell' : G \times E' \to E'$ be the left G-action on E' given by $\ell_g[g', v] = [g'g, v]$ which projects onto the canonical G-action ℓ on G/H. Since the connection form $\alpha[\chi, \mathsf{p}]$ on $G_H[U]$ is G-invariant it follows that the horizontal lift $X \mapsto X^h$ (see Section 1.1, the paragraph before eqn (1.7)) is G-equivariant in the sense that $\ell'^*_g\big(X^h\big) = (\ell^*_g X)^h$ for all $g \in G$. Moreover, G acts on the total space of the vector bundle E' as vector bundle morphisms via ℓ', whence there is a pull-back of smooth sections ψ' of E', viz $\big(\ell'^*_g\psi'\big)(x) := \ell'_{g^{-1}}\big(\psi'(gx)\big)$ for all $x \in G/H$ and some $g \in G$. Now it is straight-forward to check that ∇' is a *G-invariant covariant derivative* in the sense that for all $g \in G$ we have $\big(\ell'^*_g \nabla'_X \psi'\big) = \nabla'_{\ell^x_g(X)}\big(\ell'^*_g(\psi')\big)$ for all $g \in G$, vector fields X on G/H, and all smooth section ψ of E'.

Proposition 2.7. *Let G, H, U, χ, V, ρ as above. Suppose that the Atiyah class $c_{G,H,U,\chi}$ vanishes, and let $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ be a H-equivariant linear map extending $T_e\chi : \mathfrak{h} \to \mathfrak{u}$. Then there exists a G-invariant covariant derivative ∇ in the vector bundle E such that for all vector fields X on G/H and smooth sections $\psi \in \Gamma^\infty(G/H, E)$*

$$\Phi_V \circ \big(\nabla_X \psi\big) = \nabla'_X \Big(\Phi_V \circ \psi\Big). \tag{2.82}$$

Moreover, let $\tilde{X}$ any H-invariant lift of the vector field X to G, i.e. we have $T_g\pi\big(\tilde{X}(g)\big) = X\big(\pi(g)\big)$ for all $g \in G$ and $R^*_h\tilde{X} = \tilde{X}$ for all $H \in H$. Then there is the formula

$$\nabla_X \psi\big(\pi(g)\big) = \Big[g, \big(\tilde{X}(\hat{\psi})\big)(g) + \dot{\rho}\big(\mathsf{p}\big(T_g L_{g^{-1}}(\tilde{X}(g))\big)\big)\big(\hat{\psi}(g)\big)\Big] \tag{2.83}$$

for all $g \in G$ which does not depend on the lift $X \mapsto \tilde{X}$.

In general the lift $\tilde{X}$ will NOT be G-equivariant in the sense that the vector fields $L^*_{g_0}\tilde{X}$ and $\widetilde{\ell^*_{g_0}X}$ are equal for all $g_0 \in G$. However, the formula (2.83) does not depend on the lift $X \mapsto \tilde{X}$. Hence there can be G-invariant covariant derivatives in associated vector bundles of $\pi : G \to G/H$ which are NOT coming from a G-invariant connection in $(G, \pi, G/H, H)$ (which is equivalent to G/H being reductive, see Section 2.5.1).

Proof: Let X be a vector field on G/H and fix an H-invariant lift $\tilde{X}$ on G. According to formula (1.7) we need to compute the horizontal lift X^h of X from G/H to the total space $G_H[U]$. We use the description of $G_H[U]$ as $G \times U$ modulo the right H-action R^χ given by $(g,u)h = (gh, \chi(h)^{-1}u)$ for all $g \in G$, $h \in H$, and $u \in U$, see eqn (2.19), i.e. by means of the surjective submersion $\kappa_\chi : G \times U \to G_H[U]$, see eqn (2.23). Let $\tilde{X}^h$ be the vector field on $G \times U$ defined by (for all $g \in G$, $u \in U$):

$$\tilde{X}^h(g,u) = \Big(\tilde{X}(g), \ -T_{e_U}R_u\big(\mathsf{p}\big(T_gL_{g^{-1}}(\tilde{X}(g))\big)\big)\Big). \tag{2.84}$$

It can be easily checked using the H-equivariance of p that the following smooth map $G \times U \to T\big(G_H[U]\big)$ is invariant by the right H-action R^χ:

$$(g,u) \mapsto T_{(g,u)}\kappa_\chi\big(\tilde{X}^h(g,u)\big) \in T_{\kappa_\chi(g,u)}\big(G_H[U]\big)$$

and defines thus a unique vector field X^h on $G_H[U]$ such that $T\kappa_\chi \circ \tilde{X}^h = X^h \circ \kappa_\chi$. Using the form (2.39) of the connection 1-form $\alpha[\chi, \mathsf{p}]$ it is quickly computed that X^h is horizontal, U-invariant, and projects onto X via the bundle projection $\tau : G_H[U] \to G/H$, i.e. $T\tau \circ X^h = X \circ \tau$, whence X^h is the horizontal lift of X with respect to $\alpha[\chi, \mathsf{p}]$. Moreover, although $\tilde{X}^h$ depends on the lift $X \mapsto \tilde{X}$ the projected vector field X^h does not depend on the lift because the restriction of p to $\mathfrak{h}$ equals $T_e\chi$. Now let $\psi \in \Gamma^\infty(G/H, E)$, and let $\hat{\psi} : G \to V$ the associated H-equivariant smooth function. It is easy to compute that

$$\widehat{\Phi_V \circ \psi}\big(\kappa_\chi(g,u)\big) = \rho(u^{-1})\big(\hat{\psi}(g)\big) \tag{2.85}$$

for al $g \in G$ and $u \in U$, and therefore

$$X^h\Big(\widehat{\Phi_V \circ \psi}\Big)\big(\kappa_\chi(g,u)\big) = \rho(u^{-1})\Big(\tilde{X}\big(\hat{\psi}\big)(g) + \dot{\rho}\big(\mathsf{p}\big(T_gL_{g^{-1}}(\tilde{X}(g))\big)\big)\big(\hat{\psi}(g)\big)\Big)$$

which gives the associated U-equivariant function of the r.h.s. of the stated equation (2.82) using eqn (1.7). Again by (2.85) it is clear that this Φ_V composed with something described by the stated equation (2.83). This proves both stated equations. It is easy to check that the r.h.s. of eqn (2.83) describes a covariant derivative as it is obviously $\mathbb{R}$-bilinear, $\mathcal{C}^\infty(G/H, \mathbb{K})$ linear in the argument X (since $\widetilde{fX} = (\pi^* f)\tilde{X}$ for all $f \in \mathcal{C}^\infty(G/H, \mathbb{K})$) and first order in ψ (since $\widehat{f\psi} = (\pi^* f)\hat{\psi}$ for all $f \in \mathcal{C}^\infty(G/H, \mathbb{K})$). The G-invariance of the covariant derivative ∇ follows from the G-invariance of the covariant derivative ∇' and the fact that Φ_V is G-equivariant and invertible. □

2.5 Examples of G-ϑ-equivariant principal U-bundles and connections

2.5.1 Reductive homogeneous spaces

Let $U = H$, $\vartheta_g = \mathrm{id}_H$ for all $g \in G$, and $\chi = \mathrm{id}_H$.

It is easy to see that the Atiyah class vanishes (hence there is a G-invariant connection in the bundle $(G, \pi, G/H)$) iff there is a H-invariant projection $\mathsf{p} : \mathfrak{g} \to \mathfrak{h}$ (which is idempotent since $\mathsf{p}(\eta) = \eta$ for all $\eta \in \mathfrak{h}$), hence iff there is an H-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ which is a complement of $\mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Recall that homogeneous spaces G/H for which such an H-invariant complement $\mathfrak{m}$ to the subalgebra $\mathfrak{h}$ exists are called *reductive homogeneous spaces*, see e.g. [22, p.190]. It is well-known that for compact G all the homogeneous spaces are reductive.

Note that for a reductive homogeneous space the Atiyah class vanishes for any Lie group U and any smooth homomorphism $\chi : H \to U$ (again $\vartheta_g = \mathrm{id}_U$ for all $g \in G$): It suffices to extend the linear map $T_e\chi : \mathfrak{h} \to \mathfrak{u}$ on all of $\mathfrak{g}$ by setting it equal to zero on $\mathfrak{m}$. Hence invariant connections always exist in this case, see [36, p.10, Cor. 3].

2.5.2 Coadjoint orbits

Let $U = U(1)$ the circle group.

We shall first consider the case where ϑ is trivial, i.e.$\vartheta_g = \mathrm{id}_H$ for all $g \in G$. Let $\chi : H \to U(1)$ be a smooth homomorphism of Lie groups. Then the Atiyah class vanishes iff the linear map $T_e\chi : \mathfrak{h} \to \mathfrak{u}(1) \cong \mathbb{R}$ extends to a H-invariant linear map $\mathsf{p} : \mathfrak{g} \to \mathbb{R}$, i.e. iff $\mathsf{p} \in \mathfrak{g}^*$ and $\mathsf{p}(\eta) = T_e\chi(\eta)$ for all $\eta \in \mathfrak{h}$ and

$$\forall\ h \in H : \quad \mathsf{p} \circ Ad_h = \mathsf{p} \ \text{ iff } \ H \subset G_{\mathsf{p}} := \{g \in G \mid Ad^*_g(\mathsf{p}) = \mathsf{p}\}.$$

Clearly G_p is the isotropy group at p of the *coadjoint action* of G on the dual of its Lie algebra, $\mathfrak{g}^*$. Recall the *coadjoint orbit* $\mathcal{O}^G_{\mathsf{p}}$ passing through p defined by

$$\mathcal{O}^G_{\mathsf{p}} := \{Ad^*_g(\mathsf{p}) \in \mathfrak{g}^* \mid g \in G\} \cong G/G_{\mathsf{p}}.$$

So if the Atiyah class vanishes, the homogeneous space G/H is a G-equivariant fibre bundle over some coadjoint orbit G/G_{p} (where the obvious projection is given by $\tau(gH) = gG_{\mathsf{p}}$). In case $H = G_{\mathsf{p}}$ the coadjoint orbit is called *integral.* Note also that in case the restriction of p to $\mathfrak{h}$ the homomorphism χ is always surjective. For further information on this matter, see e.g. [24], [16] or [37].

Returning to the general case it is not hard to see that the group of all smooth automorphisms of $U(1)$ has only two elements: the identity map, and the map sending each element of the circle to its inverse. Hence for any nontrivial ϑ it follows that the Lie group G is disconnected. Moreover $\dot{\vartheta}$ necessarily vanishes in this case.

Example 2.8. Let $G = O(1,3) \times \mathbb{R}^4$ the physicist's Poincaré group where $O(1,3)$ denotes the orthogonal group of $\mathbb{R}^4$ equipped with the nondegenerate symmetric bilinear form $g(x,y) = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$ (the so-called Minkowski metric, where $x, y \in \mathbb{R}^4$, written $x = (x^0, x^1, x^2, x^3)^T$). G is the semidirect product of $O(1,3)$ and $\mathbb{R}^4$. G is known to have 4 connected components, represented by the 4 elements $e = (\mathrm{diag}(1,1,1,1), 0)$, $P = (\mathrm{diag}(1,-1,-1,-1), 0)$ ('parity'), $T = (\mathrm{diag}(-1,1,1,1), 0)$ ('time reversal'), and $PT = (\mathrm{diag}(-1,-1,-1,-1), 0)$. The factor group of G modulo its identity component is thus isomorphic to the finite group $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by the classes of the above transformations. Let $\epsilon : G \to \mathbb{Z}_2 = \{1, -1\}$ be the homomorphism of Lie groups sending each $g \in G$ first to its class modulo the identity component, and then using the map assigning to e and to P the value 1, and to T and PT the value -1. Now set U equal to the circle group $U(1)$, and define $\vartheta : G \times U \to U$ by

$$\vartheta_g(u) = u^{\epsilon(g)}. \tag{2.86}$$

Among the coadjoint orbits of the Poincaré group started by J.-M. Souriau and others there is for instance one corresponding to a massive particle with spin which has two connected components for which the symplectic 2-form used in mathematical physics is NOT equal to the usual G-invariant Kirillov-Kostant-Souriau 2-form (see e.g. eqn (3.8)) but differs by a sign on one of the components, see e.g. [37, p.121]: this is due to the fact that physicists want time reversal to be an *anti-unitary* map in Hilbert space when quantized, which corresponds to an *anti-symplectic transformation* on the orbit.

2.5.3 G-invariant linear connections in the tangent bundle of G/H

Let $U = GL(\mathfrak{g}/\mathfrak{h})$, $\vartheta_g = \mathrm{id}_U$ for all $g \in G$, and $\chi : H \to U$ given by $\chi(h) = Ad'_h$, see (2.6).

Consider the principal $GL(\mathfrak{g}/\mathfrak{h})$-bundle $G_H[U]$ over G/H. Since the group of all linear maps $GL(\mathfrak{g}/\mathfrak{h})$ acts linearly on $\mathfrak{g}/\mathfrak{h}$ there is an associated vector bundle $G_H[U][\mathfrak{g}/\mathfrak{h}]$. By means of the isomorphism (1.3) –which is clearly G-equivariant– the associated vector bundle $(G_H[U])_U[\mathfrak{g}/\mathfrak{h}]$ is isomorphic to the associated vector bundle $G_H[\mathfrak{g}/\mathfrak{h}]$ which in turn is isomorphic to the tangent bundle of G/H. Vanishing Atiyah class of the above situation is equivalent to having a G-invariant connection in the G-equivariant principal U-bundle $G_H[U]$ which will yield a *G-invariant covariant derivative* ∇ *in the tangent bundle of* G/H according to formula (1.7). Note also that the *torsion* $\mathrm{Tor}_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ for all vector fields X, Y on G/H of ∇ is a G-invariant tensor field. Hence the modified covariant derivative

$$\nabla'_X Y = \nabla_X Y - \frac{1}{2}\mathrm{Tor}_\nabla(X,Y)$$

is still G-invariant and torsion-free. Note that there is the following isomorphism $G_H[U]$ to the *bundle of all linear frames*, $P^1(G/H)$, of G/H: choose a base $e_1,\ldots,e_m$ of $\mathfrak{g}/\mathfrak{h}$, and define $\Phi : G_H[U] \to P^1(G/H)$ by (for all $g \in G$, $u \in GL(\mathfrak{g}/\mathfrak{h})$)

$$\Phi : [g,u] \mapsto \Big(T_o\ell_g\big(\pi'\big(u(e_1)\big)\big),\ldots,T_o\ell_g\big(\pi'\big(u(e_m)\big)\big)\Big),$$

and we have for all $\tilde{u} \in U$ $\Phi([g,u]\tilde{u}) = \Phi([g,u])\theta(\tilde{u})$ where $\theta(\tilde{u})$ is the matrix of $\tilde{u}$ with respect to the base $e_1,\ldots,e_m$. Clearly $\theta : GL(\mathfrak{g}/\mathfrak{h}) \to GL(m,\mathbb{R})$ is a smooth isomorphism of Lie groups, and hence Φ is an isomorphism of principal fibre bundles over G/H.

When smooth Lie group cohomology is replaced by the Chevalley-Eilenberg cohomology of the Lie subalgebra $\mathfrak{h}$, see Subsubsection 2.4.4, then the above case appeared in the literature before, see the work by Nguyen-van Hai, [31, p.46, eqs (10)-(13), Prop.3] for general Lie algebra inclusions, [29], [30] for foliations, [5], [6] for foliations of coisotropic submanifolds, and [7] for general Lie algebra inclusions $\mathfrak{h} \subset \mathfrak{g}$.

2.6 Generalisation of Infinitesimal Coadjoint Orbits

We have seen in the previous Sections essentially by Wang's Theorem how to associate G-ϑ equivariant connections in principal U-bundles over a *given* homogeneous space G/H by certain linear maps $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$. One may consider the following 'inverse problem': *given* an automorphic left G-action ϑ on U and *given* an arbitrary linear map $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$, is there a closed subgroup H of G and a crossed homomorphism $\chi : H \to U$ such that the conditions (2.36) and (2.37) of Proposition 2.3 are satisfied? In case G arbitrary, $U = U(1)$, and ϑ trivial the question is answered by any integral coadjoint orbit of G: since $\mathfrak{u}(1) \cong \mathbb{R}$ the linear maps p in question are thus elements of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. The subgroup H is defined as the *isotropy subgroup of* p, see Subsection 2.5.2. For the general situation I do not know how to do this for Lie groups. However, for the infinitesimal situation (dealt with in Subsection 2.4.4) where $\mathfrak{g}$, $\mathfrak{u}$ are two Lie algebras, $\ddot{\vartheta} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{u})$ is a derivational Lie algebra representation (for example $\ddot{\vartheta} = 0$), see eqs (2.64), and $\mathsf{p} : \mathfrak{g} \to \mathfrak{u}$ is an arbitrary linear map, the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ can –roughly speaking– be defined as the 'subalgebra of all elements of $\mathfrak{g}$ fixing p which defines the potential connection'. More precisely, define

(2.87)

$$\mathfrak{g}_{\mathsf{p}} \;:=\; \{\eta \in \mathfrak{g} \mid \forall\, \xi \in \mathfrak{g} :\; [\mathsf{p}(\eta),\mathsf{p}(\xi)] - \mathsf{p}([\eta,\xi]) + \ddot{\vartheta}_\eta\big(\mathsf{p}(\xi)\big) - \ddot{\vartheta}_\xi\big(\mathsf{p}(\eta)\big) = 0\}.$$

By an elementary computation (which is lengthy in case $\ddot{\vartheta} \neq 0$) using only the Jacobi identity of the occurring Lie brackets and the defining equations (2.64) of a derivational Lie algebra representation we get the following result generalizing the isotropy algebra $\mathfrak{g}_{\mathsf{p}}$ of $\mathsf{p} \in \mathfrak{g}^*$ for an infinitesimal coadjoint orbit:

Proposition 2.9. *With the above hypotheses:*

1. $\mathfrak{g}_{\mathsf{p}}$ *is a Lie subalgebra of* $\mathfrak{g}$.
2. *The restriction* $\dot{\chi}$ *of* p *to* $\mathfrak{g}_{\mathsf{p}}$ *defines an infinitesimal crossed morphism of Lie algebras* $\mathfrak{g}_{\mathsf{p}} \to \mathfrak{u}$*, see eqn (2.66), whence* p *satisfies eqn (2.37).*
3. p *is an infinitesimal* $\mathfrak{g}$-$\ddot{\vartheta}$*-equivariant connection in the sense of Subsection 2.4.4, i.e. for which* $\varphi[\dot{\chi}, \mathsf{p}] = 0$*, see eqn (2.70).*

Warning: note that the definition of the subalgebra $\mathfrak{g}_{\mathsf{p}}$ of $\mathfrak{g}$ given in eqn (2.87) does not work to ensure the statement of the preceding Proposition in the important particular case where $\mathfrak{u} = \mathbf{Hom}_{\mathbb{K}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$: and $\ddot{\vartheta} = 0$: here the Lie algebra $\mathfrak{u}$ depends on the subalgebra $\mathfrak{h}$! I do not know how to define some $\mathfrak{g}_{\mathsf{p}}$ in this case.

3 Multidifferential Operators over Homogeneous Spaces

3.1 Characterization of multi-differential operators

In this subsection we should like to state a Theorem about multi-differential operators on homogeneous spaces which is at least folklore because it is used for instance in [1] for a particular case.

Let G be a Lie group with Lie algebra $(\mathfrak{g}, [\ ,\ ])$, let H be a closed subgroup of G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, and let V_1,...,V_k, W be finite-dimensional H-modules (which are vector spaces over the field $\mathbb{K}$ which is either equal to $\mathbb{R}$ or equal to $\mathbb{C}$). Let $V_1^*, \ldots, V_k^*$ denote the dual H-modules. Form the associated vector bundles $E_1 = G_H[V_1]$,..., $E_k = G_H[V_k]$, $F = G_H[W]$ (with respect to the principal H-bundle $(G, \pi, G/H, H)$. We are interested in suitable description of the space of k-multi-differential operators $\mathbf{Diff}_{G/H}(E_1, \ldots, E_k; F)$.

In order to state the theorem note that any finite-dimensional H-module is also a $\mathfrak{h}$-module, and hence a left module for the universal envelopping algebra $\mathsf{U}(\mathfrak{h})$ via $(\eta_1 \eta_k)v = \eta_1(\eta_2(\cdots(\eta_k v)\cdots)$ for all $\eta_1, \ldots, \eta_k \in \mathfrak{h}$. Moreover the universal envelopping algebra of $\mathfrak{g}$, $\mathsf{U}(\mathfrak{g})$, is a right module of the $\mathsf{U}(\mathfrak{h})$ since the latter is a subalgebra of the former. In the following we shall denote by $\otimes$ the tensor products of vector spaces over $\mathbb{K}$ whereas $\otimes_{\mathsf{U}(\mathfrak{h})}$ denotes the tensor product with respect to the ring $\mathsf{U}(\mathfrak{h})$. Recall that universal envelopping algebras over finite-dimensional Lie algebras are filtered algebras $\mathsf{U}(\mathfrak{g}) = \bigcup_{i \in \mathbb{N}} \mathsf{U}_i(\mathfrak{g})$ where all the subspaces $\mathsf{U}_i(\mathfrak{g})$ are finite-dimensional.

Theorem 3.1. *With the above hypotheses and notations:*

1. *The* $\mathcal{C}^\infty(G/H,\mathbb{K})$*-module of all* k*-multi-differential operators over* G/H*,* $\mathbf{Diff}_{G/H}(E_1,\ldots,E_k;F)$*, is isomorphic to the* $\mathcal{C}^\infty(G/H,\mathbb{K})$*-module of all smooth 'filtered' sections of the associated vector bundle of the principal bundle* $(G,\pi,G/H,H)$ *with typical fibre*

$$W\otimes(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_1^*)\otimes\cdots\otimes(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_k^*). \tag{3.1}$$

   where the Lie group H *acts on this vector space diagonally on the tensor factors via the given action on* V_1^**,...,* V_k^**,* W*, via the adjoint action* Ad_h *on* $\mathfrak{g}$ *which give a unique action (also denoted by* Ad_h*) on* $\mathsf{U}(\mathfrak{g})$ *(and on* $\mathsf{U}(\mathfrak{h})$*) preserving the bialgebra structure.*

2. *The* $\mathbb{K}$*-vector space of all* G*-invariant* k*-multi-differential operators over* G/H *is isomorphic to the subspace of* H*-invariants*

$$\Big(W\otimes(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_1^*)\otimes\cdots\otimes(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_k^*)\Big)^H. \tag{3.2}$$

3. *In the particular case where all the* H*-modules* $V_1,\ldots,V_k,W$ *are equal to the trivial module* $\mathbb{K}$ *the above typical fibre (3.1) reduces to*

$$\left(\frac{\mathsf{U}(\mathfrak{g})}{\mathsf{U}(\mathfrak{g})\mathfrak{h}}\right)^{\otimes k}. \tag{3.3}$$

Proof: **1**. Write A for the associative commutative unital $\mathbb{K}$-algebra $\mathcal{C}^\infty(G/H,\mathbb{K})$, and B for $\mathcal{C}^\infty(G/H,\mathbb{K})$. Clearly the pull-back $\pi^*: A\to B$ is an injection of $\mathbb{K}$-algebras on the subalgebra of all right H-invariants in B. According to Proposition 1.2 we have in the particular case of the principal H-bundle $(G,\pi,G/H,H)$ over G/H the isomorphism of A-modules

$$\left(\frac{\mathbf{Diff}_G(G\times V_1,\ldots,G\times V_k;G\times W)}{\mathbf{K}_1+\cdots+\mathbf{K}_k}\right)^H\cong\mathbf{Diff}_{G/H}(E_1,\ldots,E_k;F).$$

It is not hard to see, using Proposition 1.1, that there is an isomorphism of B-modules

$$\mathcal{C}^\infty(G,\mathbb{K})\otimes W\otimes\big(\mathsf{U}(\mathfrak{g})\otimes V_1^*\big)\otimes\cdots\otimes\big(\mathsf{U}(\mathfrak{g})\otimes V_k^*\big)\cong\mathbf{Diff}_G(G\times V_1,\ldots,G\times V_k;G\times W).$$

which is the following map

$$\begin{aligned}&f\otimes w\otimes\mathsf{u}_1\otimes v_1^*\otimes\cdots\otimes\mathsf{u}_k\otimes v_k^*\\&\quad\mapsto\quad\Big((\psi_1',\ldots,\psi_k')\mapsto fw\big(\mathsf{u}_1^+(\langle v_1^*,\psi_1'\rangle)\big)\cdots\big(\mathsf{u}_k^+(\langle v_k^*,\psi_k'\rangle)\big)\Big).\end{aligned}$$

where $f \in \mathcal{C}^\infty(G,\mathbb{K})$, $w \in W$, $\mathsf{u}_1,\ldots,\mathsf{u}_k \in \mathsf{U}(\mathfrak{g})$, $v_1^* \in V_1^*$,..., $v_k^* \in V_k^*$, $\psi_1' \in \mathcal{C}^\infty(G,V_1)$,...,$\psi_k' \in \mathcal{C}^\infty(G,V_k)$. Note that the H-action on multi-differential operators, see (1.14) is transferred by the above isomorphism to

$$\begin{aligned} & h.\Big(f\otimes w\otimes \mathsf{u}_1\otimes v_1^*\otimes\cdots\otimes \mathsf{u}_k\otimes v_k^*\Big) \\ & \quad = \quad (f\circ R_h)\otimes\rho(h)(w)\otimes Ad_h(\mathsf{u}_1)\otimes\rho_1^*(h)\big(v_1^*\big)\otimes\cdots\otimes Ad_h(\mathsf{u}_k)\otimes\rho_k^*(h)\big(v_k^*\big). \end{aligned}$$

We shall compute the H-(and B-)submodules $\mathbf{K}_1$,..., $\mathbf{K}_k$, see eqn (1.15). Using the above map, the submodule $\mathbf{K}_j$ is spanned by elements of the following form

$$f\otimes w\otimes \mathsf{u}_1\otimes v_1^*\otimes\cdots\otimes\big(\mathsf{u}_j\eta\otimes v_j^* - \mathsf{u}_j\otimes\dot\rho_j^*(\eta)(v_j^*)\big)\otimes\cdots\otimes \mathsf{u}_k\otimes v_k^*$$

and since $\mathsf{U}(\mathfrak{h})$ is generated by all monomials $\eta_1\cdots\eta_n$ with $\eta_1,\ldots,\eta_n\in\mathfrak{h}$ we can conclude that

$$\begin{aligned} & \left(\frac{\mathbf{Diff}_G(G\times V_1,\ldots,G\times V_k;G\times W)}{\mathbf{K}_1+\cdots+\mathbf{K}_k}\right)\cong \\ & \qquad \mathcal{C}^\infty(G,\mathbb{K})\otimes W\otimes\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_1^*\big)\otimes\cdots\otimes\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_k^*\big). \end{aligned}$$

Finally, passing to the H-invariants on both sides of the above isomorphism we see that the space of multi-differential operators on G/H is given by smooth sections of a vector bundle with typical fibre (3.1).
2. Clear by eqn (2.4).
3. This is evident. □

Note that each factor in the fibre (3.1), $\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_i^*$ is a left $\mathsf{U}(\mathfrak{g})$-module, the so-called *induced module* or *Verma module* with respect to the module V_i^* of the subalgebra $\mathsf{U}(\mathfrak{h})$.

As a Corollary to the preceding Theorem 3.1 we can conclude that for any associated vector bundle $E = G_H[V]$ (where V is a finite-dimensional H-module) its rth jet prolongation ($r\in\mathbb{N}$) is isomorphic to the following associated bundle

$$J^rE_j \cong G_H\Big[\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}V_j^*\big)_r^*\Big]. \tag{3.4}$$

Exercise: The Lie bracket of vector fields on G/H obviously is a G-invariant bidifferential operator on the tangent bundle. Let $\otimes$ denote the tensor product over the ground field $\mathbb{K}$. Show that the Lie bracket is induced by an H-invariant element of $(\mathfrak{g}/\mathfrak{h})\otimes\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}(\mathfrak{g}/\mathfrak{h})^*\big)\otimes\big(\mathsf{U}(\mathfrak{g})\otimes_{\mathsf{U}(\mathfrak{h})}(\mathfrak{g}/\mathfrak{h})^*\big)$ which is obtained as follows: first project the following element of $(\mathfrak{g}/\mathfrak{h})\otimes\mathsf{U}(\mathfrak{g})\otimes\mathfrak{g}^*\otimes\mathsf{U}(\mathfrak{g})\otimes\mathfrak{g}^*$,

$$\varpi\circ[\;,\;]_{135}+\varpi_{15}\otimes(\mathrm{id}_{\mathfrak{g}})_{34}-\varpi_{13}\otimes(\mathrm{id}_{\mathfrak{g}})_{25},$$

(where the indices between 1 and 5 refer to the position of a given map in the fivefold tensor product (standard notation in Hopf algebra theory)) to $(\mathfrak{g}/\mathfrak{h})\otimes$

$\big(\mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{h})} \mathfrak{g}^*\big) \otimes \big(\mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{h})} \mathfrak{g}^*\big)$, and then check that it is in the appropriate subspace (note that by PBW $\mathsf{U}(\mathfrak{g})$ is a free, hence flat $\mathsf{U}(\mathfrak{h})$-module).

Exercise: Generalize the preceding exercise to the situation where $(\mathfrak{u}, [\ ,\ ]_\mathfrak{u})$ is another Lie algebra, $\varphi : \mathfrak{u} \to \mathfrak{g}$ and $j : \mathfrak{h} \to \mathfrak{u}$ are Lie algebra morphisms such that $\phi \circ j = i$ where $i : \mathfrak{h} \to \mathfrak{g}$ is the natural inclusion, H acts on $\mathfrak{u}$ by Lie algebra morphisms leaving invariant $j(\mathfrak{h}$ such that φ intertwines the H-action with the adjoint action, and $\varpi_\mathfrak{u} : \mathfrak{u} \to \mathfrak{u}/(j(\mathfrak{h}))$ is the natural projection. Show that the associated vector bundle $E := G_H[\mathfrak{u}/(j(\mathfrak{h}))]$ carries the structure a G-equivariant *Lie algebroid* (see e.g. [26] or [28] for definitions) where the anchor map $E \to T(G/H) \cong G_H[\mathfrak{g}/\mathfrak{h}]$ is induced by φ and the bracket is obtained as in the preceding exercise by considering in $\big(\mathfrak{u}/(j(\mathfrak{h}))\big) \otimes \mathsf{U}(\mathfrak{g}) \otimes \mathfrak{u}^* \otimes \mathsf{U}(\mathfrak{g}) \otimes \mathfrak{u}^*$ the element

$$\varpi_\mathfrak{u} \circ \big([\ ,\ ]_\mathfrak{u}\big)_{135} + \big(\varpi_\mathfrak{u}\big)_{15} \otimes \varphi_{34} - \big(\varpi_\mathfrak{u}\big)_{13} \otimes \varphi_{25}.$$

Show that all G-equivariant Lie algebroids over G/H are obtained that way (equivalence of categories).

3.2 A Theorem by Calaque, Căldăraru and Tu

In the preceding Subsection 3.1 the H-module (resp. $\mathfrak{h}$-module) $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$ turned out to be rather important. Note that this module is filtered by $\mathsf{U}_n(\mathfrak{g})/\big(\mathsf{U}(\mathfrak{g})\mathfrak{h}\big)_n$. Another filtered H-module (resp. $\mathfrak{h}$-module) is $\mathsf{S}(\mathfrak{g}/\mathfrak{h})$ where H (resp. $\mathfrak{h}$) acts via Ad' (resp. via ad') on $\mathfrak{g}/\mathfrak{h}$. One may ask the question whether these two filtered H-modules (resp. $\mathfrak{h}$-modules) are isomorphic as filtered H-modules. (resp. $\mathfrak{h}$-modules).

The question is trivial in the case when the two Hopf algebras $\mathsf{S}(\mathfrak{g})$ and $\mathsf{U}(\mathfrak{g})$ are concerned: by the Poincaré-Birkhoff-Witt (PBW) Theorem the two coalgebras $\mathsf{S}(\mathfrak{g})$ and $\mathsf{U}(\mathfrak{g})$ are isomorphic. As an isomorphism the symmetrization map $\omega : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ can be used and is known to be an isomorphism of filtered H-modules (resp. filtered $\mathfrak{h}$-modules, see e.g. [13, p.80-81]. Let $\bullet$ denote the commutative multiplication in $\mathsf{S}(\mathfrak{g})$, Δ_S be the comultiplication, and ϵ_S be the counit, whereas the multiplication in $\mathsf{U}(\mathfrak{g})$ will be denoted by $\mathsf{u}_1\mathsf{u}_2$, the comultiplication by Δ, and the counit by ϵ. In order to make computations we shall use the convolution $*$ in $\mathbf{Hom}_\mathbb{K}\big(\mathsf{S}(\mathfrak{g}), \mathsf{U}(\mathfrak{g})\big)$ using the comultiplication Δ_S and the multiplication in $\mathsf{U}(\mathfrak{g})$. Let $\mathsf{b} : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ be the linear map $i \circ \mathrm{pr}$ where pr denotes the projection $\mathsf{S}(\mathfrak{g}) \to \mathfrak{g}$, and $i_\mathfrak{g} : \mathfrak{g} \to \mathsf{U}(\mathfrak{g})$ the injection obtained by PBW. Then the solution of the differential equation

$$\frac{d\omega_t}{dt} = \mathsf{b} * \omega_t \qquad \text{with } \ \omega_0 = 1\epsilon_S$$

gives a well-defined family of coalgebra maps $t \mapsto \omega_t : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ such that $\omega_1 = \omega$, the above-mentioned symmetrization map. For $t \neq 0$ ω_t is invertible. Again for computational purposes, fix a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (where the subspace $\mathfrak{m}$ is in general not H-invariant (resp. $\mathfrak{h}$-invariant)), and for any $\xi \in \mathfrak{g}$ let $\xi_\mathfrak{m}$ its component in $\mathfrak{m}$ and $\xi_\mathfrak{h}$ its component in $\mathfrak{h}$. Let $\mathsf{b}_\mathfrak{m}$ (resp. $\mathsf{b}_\mathfrak{h}$) be the map b followed by the projection onto $\mathfrak{m}$ (resp. $\mathfrak{h}$). Consider the modified differential equation

$$\frac{d\Psi_t}{dt} = \mathsf{b}_\mathfrak{m} * \Psi_t + \Psi_t * \mathsf{b}_\mathfrak{h} \qquad \text{with} \quad \Psi_0 = 1\epsilon_S$$

It can be rapidly checked using the fact that the right multiplication with $\eta \in \mathfrak{h}$ is a coderivation of $\mathsf{S}(\mathfrak{g})$ that $t \mapsto \Psi_t$ is a family of coalgebra morphisms (which are isomorphisms for each $t \neq 0$) mapping the ideal and coideal $\mathsf{S}(\mathfrak{g}) \bullet \mathfrak{h}$ of $\mathsf{S}(\mathfrak{g})$ onto the left ideal and coideal $\mathsf{U}(\mathfrak{g})\mathfrak{h}$ of $\mathsf{U}(\mathfrak{g})$, and passes hence to the quotient to define a coalgebra isomorphism between $\mathsf{S}(\mathfrak{g}/\mathfrak{h}) \cong \mathsf{S}(\mathfrak{g})/\big(\mathsf{S}(\mathfrak{g})\bullet\mathfrak{h}\big)$ and $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$. Moreover in the reductive case, i.e. in the case where $\mathfrak{m}$ is H-invariant (resp. $\mathfrak{h}$-invariant), it follows easily that Ψ_1 and thus the induced isomorphism intertwines the H-actions (resp. the $\mathfrak{h}$-actions) proving the statement in that case which was well-known.

Recently, Calaque, Căldăraru and Tu [7] have shown the following result:

Theorem 3.2. *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0 and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then the following two statements are equivalent:*

1. *The Atiyah class of the Lie algebra inclusion (w.r.t. $\mathfrak{u} = \mathfrak{gl}(\mathfrak{g}/\mathfrak{u})$ and $\dot{\chi} = ad'$) vanishes.*
2. *The two filtered $\mathfrak{h}$-modules are isomorphic by a filtered isomorphism:*

$$\mathsf{S}(\mathfrak{g}/\mathfrak{h}) \quad \cong \quad \frac{\mathsf{U}(\mathfrak{g})}{\mathsf{U}(\mathfrak{g})\mathfrak{h}}.$$

We should like to prove the implication "**1.** $\Rightarrow$ **2.**" first of the analogous statement in smooth Lie group cohomology using differential geometry of affine connections:
Suppose that the Atiyah class $c_{G,H,GL(\mathfrak{g}/\mathfrak{h}),Ad'}$ vanishes. By Proposition 2.5 we get a G-invariant connection in the frame bundle $P^1(G/H)$ and in turn a G-invariant covariant derivative ∇ in the tangent bundle $T(G/H)$ which we can choose to be torsion-free. For any smooth manifold, let $\mathsf{S}TM$ (resp. $\mathsf{S}\big(T^*M\big)$) denote the bundle of symmetric tensors of the tangent (resp. of the cotangent) bundle. There is a differential operator of order 1, D –which depends on the covariant derivative ∇– mapping each $\Gamma^\infty\big(M, \mathsf{S}^k(T^*M)\big)$ to $\Gamma^\infty\big(M, \mathsf{S}^{k+1}(T^*M)\big)$ defined by

$$(\mathsf{D}\gamma)\big(X_1, \ldots, X_{k+1}\big) := \frac{1}{k!} \sum_{\sigma \in S_{k+1}} (\nabla_{X_{\sigma(1)}}\gamma)\big(X_{\sigma(2)}, \ldots, X_{\sigma(k+1)}\big). \tag{3.5}$$

for each $\gamma \in \Gamma^\infty\big(M, \mathsf{S}^k(T^*M)\big)$ and vector fields $X_1, \ldots, X_{k+1}$ on M. It is easy to check that D is a derivation of the commutative associative pointwise multiplication in $\Gamma^\infty\big(M, \mathsf{S}(T^*M)\big)$. Moreover, for any vector field X on M let $i_X : \Gamma^\infty\big(M, \mathsf{S}^k(T^*M)\big)$ to $\Gamma^\infty\big(M, \mathsf{S}^{k-1}(T^*M)\big)$ the usual interior product defined by $(i_X\gamma)\big(X_2, \ldots, X_k\big) = \gamma(X_1, X_2, \ldots, X_k)$ which can be canonically extended to a representation of the filtered vector bundle of commutative associative unital algebras $\Gamma^\infty\big(M, \mathsf{S}(TM)\big))$ on $\Gamma^\infty\big(M, \mathsf{S}(T^*M)\big)$ in the usual way, i.e. $i_{X_1\cdots X_r}\gamma = i_{X_1}\big(\cdots i_{X_k}(\gamma)\cdots\big)$. Recall that the *standard symbol calculus with respect to* ∇ is the linear map $\rho_S : \Gamma^\infty\big(M, \mathsf{S}(TM)\big) \to \mathbf{Diff}_M(M \times \mathbb{R}, M \times \mathbb{R})$ given by the following expression for all nonnegative integers k, $A \in \Gamma^\infty\big(M, \mathsf{S}^k(TM)\big)$, and $f \in \mathcal{C}^\infty(M, \mathbb{R})$

$$\rho_S(A)(f) = i_A\big(\mathsf{D}^k(f)\big). \tag{3.6}$$

It is well-known that the map ρ_S is a filtered isomorphism of filtered $\mathcal{C}^\infty(M, \mathbb{R})$-modules, see e.g. [4]. Moreover note that the $\mathcal{C}^\infty(M, \mathbb{R})$-module $\mathbf{Diff}_M(M \times \mathbb{R}, M \times \mathbb{R})$ is the space of all smooth sections of the filtered dual space of the jet bundle $J^\infty(M, \mathbb{R})_0$, see [23] for more informations. This means that the two vector bundles $\mathsf{S}(TM)$ and ${J^\infty(M, \mathbb{R})_0}^*$ are isomorphic by a filtered isomorphism given by ρ_S. Returning to the homogeneous space G/H: since the covariant derivative ∇ is G-invariant it follows that the isomorphism ρ_S is G-equivariant. Moreover the bundle $\mathsf{S}T(G/H)$ is isomorphic to the associated bundle $G_H[\mathsf{S}(\mathfrak{g}/\mathfrak{h})]$ whereas the graded dual of the jet bundle is given by the associated bundle $G_H[\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}]$. Using the fibre functor $E \to E_o$ from $G \cdot \mathcal{VB}_{G/H}$ we see that there is a filtered isomorphism of the filtered H-modules $\mathsf{S}(\mathfrak{g}/\mathfrak{h})$ to $\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$.

Secondly, for the pure Lie algebra case we shall make some precisions on the isomorphism: in order to get an idea we consider the formula for the covariant derivative of $\gamma \in \Gamma^\infty(M, \mathsf{S}^kT^*M)$ in the direction of a vector field X on G/H at the distinguished point $o = \pi(e)$ according to formula (2.83): let $\xi \in \mathfrak{g}$ such that $T_e\pi(\xi) = X(o)$ (we can suppose that $\xi = \tilde{X}(e)$ for some H-invariant lift $X \mapsto \tilde{X}$), and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. Note that the associated H-equivariant function $\hat{\gamma}$ is a smooth map $G \to \mathbf{Hom}_{\mathbb{K}}\big(\mathsf{S}^k(\mathfrak{g}/\mathfrak{h}), \mathbb{K}\big)$, hence

$$\begin{aligned}(\nabla_X\gamma)(o)\Big(\varpi(\xi_1)\bullet\cdots\bullet\varpi(\xi_k)\Big) &= \big(\xi^+(\hat{\gamma})\big)(e)\Big(\varpi(\xi_1)\bullet\cdots\bullet\varpi(\xi_k)\Big)\\ &- \sum_{r=1}^{k}\hat{\gamma}(e)\Big(\mathsf{p}(\xi, \varpi(\xi_r))\bullet\varpi(\xi_1)\bullet\cdots\bullet\varpi(\xi_{r-1})\bullet\varpi(\xi_{r+1})\bullet\cdots\bullet\varpi(\xi_k)\Big).\end{aligned} \tag{3.7}$$

Returning to the case of a general Lie algebra $\mathfrak{g}$ containing a subalgebra $\mathfrak{h}$: Let $j : \mathfrak{g}/\mathfrak{h} \to \mathfrak{m} \subset \mathfrak{g}$ be the inverse of the restriction of the canonical projection $\varpi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ to $\mathfrak{m}$. Define a bilinear map $\tilde{\mathsf{p}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by $\tilde{\mathsf{p}}(\xi, \xi') := j\big(\mathsf{p}(\xi)\big(\varpi(\xi')\big)\big)$. Note that $\tilde{\mathsf{p}}$ is NOT $\mathfrak{h}$-invariant, but rather

$$\big[\eta, \tilde{\mathsf{p}}(\xi, \xi')\big] - \tilde{\mathsf{p}}([\eta, \xi], \xi') - \tilde{\mathsf{p}}(\xi, [\eta, \xi']) = \big[\eta, \tilde{\mathsf{p}}(\xi, \xi')\big]_{\mathfrak{h}}$$

for all $\xi, \xi' \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$. Let $a_1, \ldots$ be a sequence of nonzero elements of $\mathbb{K}$. We translate eqn (3.7) and its symmetrization (3.5) into the definition of a 'curve of linear maps' $t \mapsto \Phi_t : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ which we write as a formal power series $\sum_{k=0}^{\infty} t^k \Phi_k$ where for each nonnegative integer k the linear map Φ_k is defined on $\mathsf{S}^k(\mathfrak{g})$ and takes its values in $\mathsf{U}(\mathfrak{g})_k$. Note that in the space of linear maps the formal power series converges in the sense that t can be replaced by any element in $\mathbb{K}$. The recursive definition goes as follows: $\Phi_0 = 1\epsilon_S$, and

$$\Phi_{k+1}(\xi_1 \bullet \cdots \bullet \xi_{k+1}) = \frac{1}{k+1} \sum_{r=1}^{k+1} \xi_r \big(\Phi_k(\xi_1 \bullet \cdots \bullet \xi_{r-1} \bullet \xi_{r+1} \bullet \cdots \bullet \xi_{k+1})\big)$$
$$-\frac{1}{k+1} \sum_{r=1}^{k+1} \sum_{\substack{s=1 \\ s\neq r}}^{k+1} \Phi_k\big(\tilde{\mathsf{p}}(\xi_r, \xi_s) \bullet \xi_1 \bullet \cdots \bullet \xi_{r-1} \bullet \xi_{r+1} \bullet \cdots \bullet \xi_{s-1} \bullet \xi_{s+1} \bullet \cdots \bullet \xi_{k+1}\big).$$

First, since obviously $\Phi_k(\xi_1 \bullet \cdots \bullet \xi_k)$ is a nonzero multiple of $\xi_1 \cdots \xi_k$ modulo $\mathsf{U}(\mathfrak{g})_{k-1}$ it follows by PBW that Φ is a linear bijection. Let $\Pi : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}$ denote the canonical projection. It can be shown by a lengthy, but elementary induction over k that **i)** the map $\Pi \circ \Phi_k$ vanishes if one of the arguments is in $\mathfrak{h}$ and that **ii)** the map $\Pi \circ \Phi_k$ intertwines the $\mathfrak{h}$-actions (where the induction step has to be done for both statements at the same time). Hence Φ sends the (co)ideal $\mathsf{S}(\mathfrak{g})\bullet\mathfrak{h}$ of $\mathsf{S}(\mathfrak{g})$ into the left ideal and coideal $\mathsf{U}(\mathfrak{g})\mathfrak{h}$ of $\mathsf{U}(\mathfrak{g})$, and the above filtration argument plus an induction shows that this is onto. Hence there is an induced isomorphism of $\mathfrak{h}$-modules $\mathsf{S}(\mathfrak{g}/\mathfrak{h}) \to \mathsf{U}(\mathfrak{g})/(\mathsf{U}(\mathfrak{g})\mathfrak{h})$ which is an isomorphism of coalgebras for $t \neq 0$. Alternatively, in order to avoid fiddling around tedious combinatorics, one may consider the following differential equation: let r denote the linear map $\mathsf{S}^2(\mathfrak{g}) \to \mathfrak{g}$ defined by $\mathsf{r}(\xi_1 \bullet \xi_2) = \tilde{\mathsf{p}}(\xi_1, \xi_2) + \tilde{\mathsf{p}}(\xi_2, \xi_1)$. Moreover let $\tilde{*}$ denote the convolution of linear maps from $\mathsf{S}(\mathfrak{g})$ to $\mathsf{S}(\mathfrak{g})$ with respect to $\bullet$ and Δ_S. Then

$$\frac{d\Phi_t}{dt} = \mathsf{b} * \Phi_t - \Phi_t \circ (\mathsf{r}\tilde{*}\mathrm{id}_{\mathsf{S}(\mathfrak{g})}) \qquad \text{with} \quad \Psi_0 = 1\epsilon_S$$

defines the above 'curve'. Note that $\mathsf{r}\tilde{*}\mathrm{id}_{\mathsf{S}(\mathfrak{g})}$ is a coderivation of the coalgebra $\mathsf{S}(\mathfrak{g})$. The identities can be proved by showing that they satisfy certain differential equations with initial condition 0, hence they hold for all t by uniqueness of the solution. □

3.3 Invariant star-products on certain coadjoint orbits

Let G be a Lie group, $(\mathfrak{g}, [\ ,\ ])$ be its Lie algebra, and $\mathsf{p} \in \mathfrak{g}^*$. Let $\mathcal{O}^G_{\mathsf{p}}$ the coadjoint orbit $\{Ad^*_g(\mathsf{p}) \mid g \in G\}$, and G/H its associated homogeneous space, i.e. H is the isotropy subgroup $H = G_{\mathsf{p}} = \{g \in g \mid Ad^*_g(\mathsf{p}) = \nu\}$. The orbit and G/H are well-known to be *Hamiltonian G-spaces*, i.e. there is a canonical G-invariant symplectic 2-form, the Kirillov-Kostant-Souriau form given by

$$\omega_{\pi(g)}\big([g, \pi'(\xi)], [g, \pi'(\xi')]\big) = \langle \mathsf{p}, [\xi, \xi'] \rangle \qquad (3.8)$$

for all $g \in G$, $\xi, \xi' \in \mathfrak{g}$, see e.g. [16] for details, and a momentum mapping $J : G/H \to \mathfrak{g}^*$ given by $J\big(\pi(g)\big) = Ad^*_g(\mathsf{p})$. Recall that a *star-product* on a symplectic or more general Poisson manifold is a formal associative deformation of the associative commutative unital algebra of all smooth $\mathbb{K}$-valued functions on the manifold such that all the bilinear maps are bidifferential operators and that the first order commutator is proportional to the Poisson bracket, see e.g. [3], [15] or [35].

Proposition 3.3. *With the above notations: suppose that the Atiyah class with respect to the induced adjoint representation $Ad' : H \to Gl(\mathfrak{g}/\mathfrak{h})$, $c_{G,H,Gl(\mathfrak{g}/\mathfrak{h}),Ad'}$, vanishes. Then there is a G-invariant star-product $*$ on G/H.*

Proof: It follows that there is a G-invariant torsion-free covariant derivative in the tangent bundle of G/H. Using the Heß-Lichnerowicz-Tondeur trick, see e.g. [35, p.454], there is a G-invariant torsion-free symplectic connection in the tangent bundle. Using a Theorem by B.Fedosov, see [15, 180-183] there is G-invariant star-product on G/H. □.

Note that the series of bidifferential operators of the star-product is an element of $\Big(\mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h} \otimes \mathsf{U}(\mathfrak{g})/\mathsf{U}(\mathfrak{g})\mathfrak{h}\Big)^H[[\lambda]]$ (which has been noted in [1]). For compact coadjoint orbits there is Karabegov's construction in [19] whereas for certain $\mathbb{Z}$-graded coadjoint orbits Alekseev and Lakhowska constructed a star-product via the Shapovalov trace, see [1]. Pikulin and Tevelev classified those nilpotent orbits of reductive Lie groups for which the Atiyah class vanishes and only found a small class, see [33] for details.

References

[1] Alekseev, A., Lakhowska, A.: *Invariant $*$-products on coadjoint orbits and the Shapovalov pairing.* Comment. Math. Helv. **80** (2005), no. 4, 795-810.

[2] Atiyah, M.F.: *Complex analytic connections in fibre bundles.* Trans. Amer. Math. Soc. **85** (1957), 181-207.

[3] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization.* Annals of Physics **111** (1978), part I: 61-110, part II: 111-151.

[4] Bordemann, M., Neumaier, N., Pflaum, M., Waldmann, S.: *On representations of star product algebras over cotangent spaces on Hermitian line bundles.* J.Funct.Analysis **199** (2003), 1-47.

[5] Bordemann, M.: *(Bi)modules, morphismes et réduction des star-produits: le cas symplectique, feuilletages et obstructions.* Preprint `math.QA/0403334`, 2004.

[6] Bordemann, M.: *(Bi)Modules, morphisms, and reduction of star-products: the symplectic case, foliations, and obstructions.* Travaux mathématiques **16** (2005), 9-40.

[7] Calaque, D., Căldăraru, A., Tu, J.: PBW *for an inclusion of Lie algebras.* `arXiv:1010.0985v2 [math.QA] 9 May 2011`.

[8] Calaque, D.: *A* PBW *theorem for inclusions of (sheaves of) Lie algebroids.* `arXiv:1205.3214v1 [math.QA] 14 May 2012`.

[9] Cartan, H., Eilenberg, S.: *Homological Algebra.* Princeton University Press, 1956.

[10] Cattaneo, A., Torossian, C.: *Quantification pour les paires symetriques et diagrammes de Kontsevich.* Ann. Sci. Ec. Norm. Sup. **41** (2008), 789-854.

[11] Cattaneo, A., Torossian, C.: *Biquantization of symmetric pairs and the quantum shift.* Preprint `arXiv:1105.5973`, 2011.

[12] Zhuo Chen, Mathieu Stiénon, Ping Xu: *From Atiyah classes to homotopy Leibniz algebras.* Preprint `arXiv:1204.1075v1` 4 April 2012.

[13] Dixmier, J.: *Algèbres enveloppantes.* Gauthier-Villars, Paris 1974.

[14] Duflo, M.: *Open problems in representation theory of Lie groups.* In: ed. T.Oshima *Analysis on homogeneous spaces*, Proceedings of a conference in Katata, Japan, 1986, University of Tokyo, 1987, pp. 1-5.

[15] Fedosov, B.: *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin, 1996.

[16] V.Guillemin, S.Sternberg: *Symplectic Techniques in Physics.* 1984.

[17] Gutt, S.: *An explicit $*$-product on the cotangent bundle of a Lie group.* Lett.Math.Phys. **7** (1983), 249-258.

[18] Hochschild, G.: *The Automorphism Group of a Lie Group.* Trans. AMS **75** (1952), 209-216.

[19] Karabegov, A.V.: *Berezin's quantization on flag manifolds and spherical modules.* Trans. Amer. Math. Soc. **350** (1998), no. 4, 1467-1479.

[20] Kashiwara, M., Shapira, P.: *Categories and Sheaves.* Springer, Heidelberg, 2006.

[21] Kobayashi, S., Nomizu, K.: *Foundations of Differential Geometry.* Vol I. Wiley, New York, 1963.

[22] Kobayashi, S., Nomizu, K.: *Foundations of Differential Geometry.* Vol II. Wiley, New York, 1969.

[23] Kolář, I., Michor, P., Slovák, J.: *Natural Operations in Differential Geometry.* Springer, Berlin, 1993.

[24] Kostant, B.: *Quantization and Unitary Representations.* In ed. C.T.TAAM '*Lectures in Modern Ananlysis and Applications III*', Lectures Notes in Mathematics **170**, Springer, Berlin, 1970, 87-208.

[25] Loday, J.L.: *Cyclic Homology.* Springer, Berlin, 1992.

[26] Mackenzie, K.: *General Theory of Lie Groupoids and Lie Algebroids.* London Mathematical Society Lecture Note Series: **213**, Cambridge University Press, Cambridge, UK, 2005.

[27] Mac Lane, S.: *Categories for the Working Mathematician.* 2nd ed., Springer, New York, 1998.

[28] Moerdijk, I., Mrčun, J.: *Lie groupoids, sheaves and cohomology.* In: Gutt, S., Rawnsley, J., Sternheimer, D.:*Poisson Geometry, Deformation Quantisation and Group Representations*, London Math.Soc. Lect. Notes **323**, Cambridge University Press, Cambridge, UK, 2005, 145-272.

[29] Molino, P.: *Propriétés cohomologiques et propriétés topologiques des feuilletages à connexion transverse projetable.* Topology **12** (1973), 317-325.

[30] Molino, P.: *La classe d'Atiyah d'un feuilletage comme cocycle de déformation infinitésimale.* C.R.Acad.Sc. Paris **278** (1974), 719-721.

[31] Nguyen-van Hai *Relations entre les diverses obstructions relatives à l'existence d'une connexion linéaire invariante sur un espace homogène.* C. R. Acad. Sci. Paris **260** (1965), 45-48.

[32] Palais, R. S.: *On the Existence of Slices for Actions of non-compact Lie Groups.* Ann. Math. **73** (1961), 295-323.

[33] Pikulin, S.V., Tevelev, E.A.: *Invariant linear connections on homogeneous symplectic varieties.* Transformation Groups **6** (2001), 193-198.

[34] Wagemann, F., Wockel, C.: *A Cocycle Model for for Topological and Lie Group Cohomology.* Preprint `arXiv:1110.3304v2 [math.AT]`, October 2011.

[35] Waldmann, S.: *Poisson-Geometrie und Deformationsquantisierung.* Springer, Berlin, 2007.

[36] Wang, H.-C.: *On Invariant Connections Over A Principal Fibre Bundle.* Nagoya Math. J. **13** (1958), 1-19.

[37] Woodhouse, N.M.J.: *Geometric Quantization*. Clarendon Press, Oxford, UK, 1992.

Martin Bordemann
Laboratoire de Mathématiques, Informatique et Applications
Université de Haute-Alsace Mulhouse
4, Rue des Frères Lumière
F.68093 Mulhouse
France
email: Martin.Bordemann@uha.fr