

A very short presentation of deformation quantization,
some of its developments in the past two decades,
and conjectural perspectives¹

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Abstract

Deformation quantization is the main paradigm for Flato’s “deformation philosophy” on how to interpret the emergence of new physical theories. It gives a framework in which quantization can be understood as a deformation of the classical (commutative) composition law of observables, functions on phase space (manifolds with a Poisson bracket). We sketch its formulation and significant examples showing the essence of deformation quantization, and its relations with usual quantization. We then indicate some developments and avatars in the past two decades, during Neumaier’s active scientific life. We end with the presentation of a multifaceted framework in which Anti de Sitter (space and/or symmetry) would be quantized, with conjectural implications in cosmology and to a deformations-based possible space-time origin of elementary particle symmetries.

1 Introduction: the deformation philosophy

However seducing the idea may be, the notion of “Theory of Everything” is to me unphysical. In physics, sometimes knowingly but often not (because one simply ignores a more elaborate reality that has yet to be discovered), one makes approximations in order to have as manageable a theory (or model) as possible, or simply to try and describe the reality known at the time. In other words, physical theories have their domain of applicability defined e.g. by the relevant distances, velocities, energies, etc. involved. However in physics the passages from one domain (of distances, etc.) to another do not happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict accepted

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theories. Eventually a new fundamental constant enters and the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, “contracts” to the previous formalism.

A first example of that phenomenon can be traced back to the antiquity, when it was gradually realized that the earth is not flat, and in mathematics to the nineteenth century with Riemann surface theory. However the main developments happened a century later, in particular with the seminal analytic geometry works of Kodaira and Spencer [KS58] (and their lesser known interpretation by Grothendieck [Gr61], where one can see in watermark his “EGA” that started a couple of years later). These deep geometric works were in some sense “linearized” in the theory of deformations of algebras by Gerstenhaber [Ge64]. The realization that deformations are fundamental in the development of physics happened a couple of years later in France, when it was noticed that the passage from the Galilean invariance of Newtonian mechanics ($SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$) is deformed, in the Gerstenhaber sense [Ge64], to the Poincaré group ($SO(3,1) \cdot \mathbb{R}^4$) of special relativity. In spite of the fact that the composition law of symbols of pseudodifferential operators, essential in the Atiyah–Singer index theorem developed at that time (to the exposition of which I took part in Paris in 1963/64), was in effect a deformation of their abelian product, it took another ten years or so to develop the tools which enabled us to make explicit, rigorous and convincing, what was in the back of the mind of many: quantum mechanics is a deformation of classical mechanics. That developed into what became known as *deformation quantization* and its manifold avatars and more generally into the realization that quantization is deformation. This stumbling block being removed, the paramount importance of deformations in theoretical physics became clear [Fl82], giving rise to “Flato’s deformation philosophy”.

2 The essence of deformation quantization

For a quasi-complete (not in the topological vector spaces sense!) overview of the “state of the art” at the turn of the millennium, see e.g. [DS01], including references therein. Shorter later presentations can be found in e.g. [St08] and [St11] and references therein. To make the presentation slightly self-contained, we mention here some main points, indicating in particular that deformation quantization *is* quantization, without the sometimes Procrustean bed of a Hilbert space formulation – something that may seem heretic to those bred within the “Copenhagen interpretation” taken restrictively,

2.1 The founding papers in the 70's and around

2.1.1 Classical mechanics

In its Hamiltonian formulation, classical mechanics is based on a phase space, a symplectic or more generally Poisson (see e.g. [BFFLS]) manifold W on which a function H (the Hamiltonian) expresses the dynamics of the system considered: W is a differentiable manifold on which is defined a skewsymmetric contravariant tensor π (not necessarily nondegenerate, and which can be expressed in local coordinates as $\pi = \sum_{i,j=1}^{2n} \pi^{ij} \partial_i \wedge \partial_j$) such that $\{F, G\} = \pi(dF \wedge dG)$ (in local coordinates, $\{F, G\} = \sum_{i,j} \pi^{ij} \partial_i F \wedge \partial_j G$) is a Poisson bracket P , i.e. the bracket $\{\cdot, \cdot\}: C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ is a bilinear map which is skewsymmetric ($\{F, G\} = -\{G, F\}$) and satisfies the Jacobi identity $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$ and the Leibniz rule $\{FG, H\} = \{F, H\}G + F\{G, H\}$.

For symplectic manifolds the 2-tensor π is everywhere nondegenerate (it then has an inverse, a closed 2-form ω ; closedness is equivalent to the Jacobi identity). The simplest example is $W = \mathbb{R}^{2n}$ with coordinates (q_α, p_α) , $\alpha = 1, \dots, n$, e.g. m particles in 3-space with $n = 3m$. The motion of a particle in 3-space is invariant under the Galileo group $SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$ (space rotations, velocity translations and space-time translations, respectively). The Poisson bracket of two classical observables (functions F and G) has then the well known expression $\{F, G\} = \sum_\alpha \frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial G}{\partial q_\alpha} \frac{\partial F}{\partial p_\alpha}$. The Hamiltonian is a real-valued function $H(q, p)$ on phase space and Hamilton's equations of motion for an observable F (e.g. a coordinate) are $\frac{dF}{dt} \equiv \dot{F} = \{H, F\}$. An important example of Poisson manifold that is not symplectic is the dual \mathfrak{g}^* of a Lie algebra \mathfrak{g} . Any Poisson manifold is "foliated" by symplectic leaves (in general, of variable even dimension).

2.1.2 Quantum mechanics

The idea of quantization arose around 1900 when Planck proposed the quantum hypothesis: the energy of light is not emitted continuously but in quanta proportional to its frequency. Einstein's 1905 theory of the photoelectric effect (which was the reason for which he eventually was awarded in 1923 the 1922 Nobel prize in physics) builds on that idea, as well as Bohr's 1913 model for an atom with "quantized" orbits for electrons around the nucleus. But the real beginning of quantum mechanics dates from the mid twenties when Louis de Broglie suggested in 1923 what he called "wave mechanics" based on the (somewhat schizophrenic) idea that waves and particles are two aspects of the same physical reality. The idea was shortly afterward confirmed with the discovery of electron diffraction by crystals in 1927 by Davisson and Germer, and Louis de Broglie was awarded the 1929 Nobel Prize in physics. A couple of years after de Broglie, Schrödinger,

Heisenberg, Weyl and eventually Bohr (inter alia) “translated into German”² de Broglie’s idea and developed the quantum mechanics that we know, with for observables operators in a space introduced some years before by Hilbert, and the “Copenhagen” probabilistic interpretation that comes with it – which until now a number of eminent (and less eminent) physicists are not entirely happy with.

In the traditional quantization of a classical system $(\mathbb{R}^{2n}, \{\cdot, \cdot\}, H)$ we take a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \ni \psi$ in which acts a “quantized” Hamiltonian \hat{H} , the energy levels of which are defined by an eigenvalue equation $\hat{H}\psi = \lambda\psi$. An essential ingredient is the von Neumann representation of the canonical commutation relations (CCR) for which, defining the operators $\hat{q}_\alpha(f)(q) = q_\alpha f(q)$ and $\hat{p}_\beta(f)(q) = -i\hbar \frac{\partial f(q)}{\partial q_\beta}$ for f differentiable in \mathcal{H} , we have (CCR) $[\hat{p}_\alpha, \hat{q}_\beta] = i\hbar \delta_{\alpha\beta} I$ ($\alpha, \beta = 1, \dots, n$). We say that the couple (\hat{q}, \hat{p}) “quantizes” the coordinates (q, p) . A polynomial classical Hamiltonian H is quantized once chosen an operator ordering [AW70], e.g. the (Weyl) complete symmetrization of \hat{p} and \hat{q} . In general the quantization on \mathbb{R}^{2n} of a function $H(q, p)$ with inverse Fourier transform $\tilde{H}(\xi, \eta)$ can be given by (that formula is already in Weyl [We27] when the weight is $\varpi = 1$):

$$(2.1) \quad H \mapsto \hat{H} = \Omega_\varpi(H) = \int_{\mathbb{R}^{2n}} \tilde{H}(\xi, \eta) \exp(i(\hat{p} \cdot \xi + \hat{q} \cdot \eta)/\hbar) \varpi(\xi, \eta) d^n \xi d^n \eta.$$

The weight ϖ is, in most orderings, the exponential of a second order polynomial ($\simeq \xi\eta$ or $\simeq \xi^2 \pm \eta^2$ for the main orderings).

The map Ω_1 given by (2.1) has an inverse, obtained by Wigner [Wi32] which (when the expression is defined) can be written as:

$$(2.2) \quad H = (2\pi\hbar)^{-n} \text{Tr}[\Omega_1(H) \exp((\xi \cdot \hat{p} + \eta \cdot \hat{q})/i\hbar)]$$

The commutator of two operators comes then from what is now called the Moyal bracket of the corresponding classical observables:

$$(2.3) \quad M(u_1, u_2) = \nu^{-1} \sinh(\nu P)(u_1, u_2) = P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{\nu^{2r}}{(2r+1)!} P^{2r+1}(u_1, u_2)$$

where P denotes the Poisson bracket and the “deformation parameter” is $\nu = i\frac{\hbar}{2}$ while the product of operators comes from:

$$(2.4) \quad u_1 \star_M u_2 = \exp(\nu P)(u_1, u_2) = u_1 u_2 + \sum_{r=1}^{\infty} \frac{\nu^r}{r!} P^r(u_1, u_2)$$

the two being related by $M(u_1, u_2) = \frac{1}{i\hbar}(u_1 \star_M u_2 - u_2 \star_M u_1)$. We recognize in the right-hand sides of (2.3) and (2.4) the formulas for deformations of algebras

²Compare with Goethe’s quote: *Die Mathematiker sind eine Art Franzosen. Spricht man zu ihnen, so übersetzen sie alles in ihre eigene Sprache, und so wird es alsobald etwas ganz anderes.* (Mathematicians are a kind of Frenchmen. Whenever you talk to them, they translate everything into their own language, and right away it becomes something completely different.)

(in this case, Lie with bracket P and associative with usual product of functions, resp.) in the sense of Gerstenhaber [Ge64] where the Chevalley–Eilenberg (resp. Hochschild) cochains giving the deformation are (resp.) $\frac{P^{2r+1}}{(2r+1)!}$ and $\frac{P^r}{r!}$.

2.1.3 Deformation quantization

In the early 70’s, having in mind the deformation philosophy, we started to study 1-differentiable deformations (in the sense of Gerstenhaber) of the Lie algebra of classical observables (“functions” on phase space) endowed with the Poisson bracket [FLS75]. Then Jacques Vey [Ve75], inspired by our works, showed the existence of differentiable deformations of the Poisson bracket Lie algebra $A = C^\infty(W)$ (of functions on a symplectic manifold W with vanishing 3rd Betti number b_3). Doing so he rediscovered the Moyal bracket [Mo49] and the Hochschild cohomology of the associative algebra structure on A , which turned out to follow from [HKR62], both of which (like most physicists and mathematicians at the time) he was unaware of. That in turn triggered our development of what people now call “the founding papers” [BFFLS] and deformation quantization.

In keeping with the style of this presentation in which we insist on the conceptual aspect and refer to the bibliography (and references therein) for a little more details, we shall briefly give some characteristic features, including physical examples, showing that we can indeed say in confidence that *quantization is deformation*, deformation being of course a more general notion (cf. e.g. deformations within the Lie algebra category).

Let W be a differentiable manifold (of finite, or possibly infinite, dimension). We assume given on W a Poisson bracket P .

Definition 2.1. A *star product* on W is a deformation of an associative algebra of functions, e.g. $A = C^\infty(W)$, of the form $\star = \sum_{n=0}^{\infty} \nu^n C_n$ with $C_0(u, v) = uv$, $C_1(u, v) - C_1(v, u) = 2P(u, v)$, $u, v \in A$, the C_n being bidifferential operators (locally of finite order). A star-product is *strongly closed* if it satisfies a trace condition, $\int_W (u \star v - v \star u) dx = 0$ where dx is a volume element on W .

In deformation quantization, quantization is not performed with a drastic change in the nature of observables, but understood as a deformation of the abelian composition law of these observables, piloted by the Poisson bracket. It is *more* than “a reformulation of the problem of quantizing a classical mechanical system” [DN01], it *is* quantization, in spite of what two brilliant scientists wrote recently [GW08], after very nice words on the approach (viewed mostly from a ‘stringy’ perspective): “deformation quantization is not quantization. [...] It does not lead to a natural Hilbert space H on which the deformed algebra acts.” In other words, the “original sin” of deformation quantization would be that (in the general case) we do not need the sacrosanct Hilbert space required by the Copenhagen interpretation. Nevertheless deformation quantization is in line with a prophetic general statement by Dirac [Di49], which applies to many situations in physics:

Two points of view may be mathematically equivalent, and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics.

What Dirac had then in mind is certainly the quantization of constrained systems which he developed shortly afterward [Di50]. In that case the “canonical quantization” mentioned in Section 2.1.2 cannot be used. Dirac’s “by hand” method is a typical example of quantizing symplectic manifolds (with second class constraints) and Poisson manifolds (with first class constraints) (see e.g. [Li75]).

Dirac’s statement applies even better to deformation quantization, which (see below) works for any (finite dimensional) Poisson manifold. When there are Weyl (2.1) or Wigner (2.2) maps between an algebra of functions and an algebra of operators, the two formalisms are (more or less) equivalent. If that is not the case, deformation quantization is basically what is left. Without the Hilbert space restrictive frame there may sometimes be too much freedom in deformation quantization, depending on how it is performed (e.g. with star-spectral equations), and some “auxiliary conditions” are then needed (as noted also by Gukov, private communication), but in my view that is preferable to the restrictive approach of geometric quantization, which proved to be very powerful in representation theory of Lie groups (in particular solvable) but with which only few observables could be quantized. These “auxiliary conditions” were in fact “built in” the examples treated in [BFFLS]. Indeed, these follow from the closed formulas which we obtained (in examples, with some effort and a little bit of luck) for the analog of the unitary evolution operator, the *star exponential* $\text{Exp}_*(\frac{tH}{i\hbar}) = \sum_{r \geq 0} \frac{1}{r!} (\frac{t}{i\hbar})^r H^{*r}$ (where $2\nu = i\hbar$ and H^{*r} denotes the r^{th} star power of H). It is a singular object, in particular it does not belong to the quantized algebra $(A[[\nu]], *)$ but to $(A[[\nu, \nu^{-1}]], *)$. *Spectrum and states* are given by a “spectral” (Fourier-Stieltjes in the time t) decomposition of the star-exponential.

Examples. In order to show that a star-product provides an *autonomous* quantization of a manifold M we treated in [BFFLS] a number of examples. For the harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$, with the Moyal product on \mathbb{R}^{2n} , we obtain $\text{Exp}_*(\frac{tH}{i\hbar}) = (\cos(\frac{t}{2}))^{-1} \exp(\frac{2H}{i\hbar} \tan(\frac{t}{2})) = \sum_{k=0}^{\infty} \exp(-i(k + \frac{n}{2})t) \pi_k^n$ where π_k^n can be expressed as a function of H . As expected the energy levels of H are $E_k = \hbar(k + \frac{n}{2})$. With normal ordering, $E_k = k\hbar$. While $E_0 \rightarrow \infty$ for $n \rightarrow \infty$ in Moyal ordering, $E_0 \equiv 0$ in normal ordering, thus preferred in Field Theory.

For such a formal series of formal series to be well-defined as a formal series, we need that the coefficients of the resulting powers of \hbar be finite. That requirement

is here a “built in” auxiliary condition, expressed in the closed formulas obtained, well defined as distributions in both p, q and t .

That one-parameter group (with parameter t) can be completed to what we call a *star representation* of (a 2-fold covering of) the Lie group $\mathrm{SL}(2, \mathbb{R})$, corresponding to the metaplectic representation. In retrospect we were somewhat lucky because what corresponds to the trace of operators is the integral over phase space (\mathbb{R}^{2n} here) of the corresponding functions (or distributions), which is the analog of the character of the representation. It is one of the tools which permit a comparison with usual representation theory and is often singular at the origin in irreducible representations (e.g. as shown by Harish Chandra, for semi-simple Lie groups). That requires caution in computing the star exponential. But in the case of the harmonic oscillator, the difficulty is masked by the fact that the corresponding representation of the Lie algebra $\mathfrak{sl}(2)$ generated by (p^2+q^2, p^2-q^2, pq) is integrable to a double covering of $\mathrm{SL}(2, \mathbb{R})$ and decomposes into a sum, usually denoted by $D(\frac{1}{4}) \oplus D(\frac{3}{4})$: the singularities at the origin cancel each other for the two components. This made possible the computation of the above closed formula for the star exponential of the compact generator H , and provided implicitly the required auxiliary conditions, that do not appear when trying to compute directly the “star spectrum” of H with an equation of the type $H * \pi_k = E_k \pi_k$.

Other standard examples can be quantized in an autonomous manner by choosing adapted star products, e.g. the angular momentum with spectrum $k(k + (n - 2))\hbar^2$ for the Casimir element of $\mathfrak{so}(n)$ and the hydrogen atom with $H = \frac{1}{2}p^2 - |q|^{-1}$ on $M = T^*S^3$, $E = \frac{1}{2}(k + 1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E \in \mathbb{R}^+$ for the continuous spectrum; etc.

2.2 Some of the main progresses in the 80’s

The “founding papers” created among many a strong interest in the new notion. The idea was “in the back of the mind” of most of those who dealt with quantum mechanics [one even wrote us, asking to be quoted for that reason, but we did not know how to refer to the back of the mind of that person!]. But the Hilbert space formalism created a “quantum jump” (from classical observables to operators in Hilbert space) that was hard to express. It was only when we dared the unconventional approach – may be indirectly related to de Broglie’s idea that particles and waves are two manifestations of the same physical reality – to look at quantization as a deformation of the same algebra from commutative to noncommutative, that (as Commissaire Maigret says when he discovers “whodunit”: “But of course!”) what is for us the essence of quantization became clear.

The notion of *equivalence* of star products is the standard one for deformations, a formal series of linear maps (here necessarily differential operators) intertwining two star products. By equivalence any star product can be brought to one for which the function 1 is still a unit (that follows from a general result of Gerstenhaber [GS88] on deformations leaving a subalgebra invariant). The *existence* of

star products was shown in increasing generality, first in the “founding papers”, then for symplectic manifolds with Betti number $b_3 = 0$, eventually for all symplectic manifolds in the 80’s, and later for Poisson manifolds. We shall briefly come back to these issues in the general context of Section 3.1. For some more details and references see e.g. [DS01].

Of particular interest are the notions of invariance and of covariance of star products, which occupy a significant part of the first “founding paper” and shortly afterward gave rise to the interesting notion of “star representations” (without operators) of Lie groups and algebras.

2.3 “Metamorphoses” in the 80’s, quantum groups and noncommutative geometry

Starting from different premises, important developments appeared in the 80’s which, a posteriori and though additional notions are involved, can be considered as metamorphoses or avatars of deformation quantization. For the record we mention here very briefly the main two.

The first is the notion of what are now called quantized enveloping algebras of Lie groups, which appeared in works by the Faddeev school in Leningrad around 1980, in relation with the inverse problem for two-dimensional integrable models in quantum field theory. A few years later Drinfeld made the notion more systematic as deformations of Hopf algebras and popularized it under the name “quantum groups” together with their “dual” notion of star products on functions on Lie groups. (It is a dual in the sense of topological vector spaces duality, as shown by us in the 90’s, see e.g. the review part of [BGG].) Several books have been dedicated to the subject and its applications, with thousands of references.

A second major development is the advent, from the beginning of the 80’s and motivated by his seminal works on operator algebras in the 70’s, of Alain Connes’ noncommutative geometry [Co94] of which (as we showed in a joint paper) star products algebras (having a trace) constitute another example. That is since then a very active frontier domain of mathematics with a variety of applications, including in physics, and a dedicated journal.

3 Some highlights of the last two decades

We mention here very briefly a few highlights, in particular those related to the published works of Neumaier (in good journals, between 1998 and 2010). Again this section is not meant to be exhaustive. Slightly less concise presentations with more references can be found in [DS01] and later reviews.

3.1 Existence and Classification (symplectic and Poisson manifolds)

The question of existence of star products was first solved in the 80's for symplectic manifolds [DL83], based on the idea of gluing local Moyal star-products defined on Darboux charts, using algebraic and cohomological tools. Together with their classification, that became understood geometrically only in the 90's.

For symplectic manifolds the idea is to “glue” together Moyal products on symplectic (Darboux) charts: Since the Moyal product on a chart is unique (up to equivalence) that is always possible on the intersection of two charts, but problems occur on the intersection of 3 charts, except when $b_3 = 0$ (as we mentioned in Section 2.3). The idea (which has received a number of formulations, some quite sophisticated) is essentially to use, instead of the original manifold W , a bundle of Weyl algebras (CCR) on W , obtain a global section and project it on W . That was used in two (related) forms. A “local” form (on sizeable charts) was developed by a Japanese group [OMY] and shortly afterward an alternative to the original construction of [DL83] was developed by its authors [DL92].

In 1985, motivated by index theory and independently of the previous proofs, Fedosov (who unfortunately died recently) announced a purely geometrical construction of star-products on a symplectic manifold, also based on Weyl algebras, but that second method was noticed only when a complete version was published in an international journal [Fe94]. The beautiful algorithmic construction of Fedosov (in which the Maurer-Cartan equation has a crucial role) not only provides a novel proof of existence, it also gives a much better understanding of deformation quantization, paving the way to further major developments. The construction and its context were developed in a book by Fedosov and outlined in many papers, including in our review [DS01] where many references can be found.

The relations between the “Russian” (Fedosov) approach and that of the “Belgian” team were creatively expressed in his own language by a famous Belgian mathematician (with Russian wife) in an interesting paper published in Gelfand's journal [De95]. Though (unfairly) it ignores the above mentioned Japanese approach, the paper deserves to be better known, in particular in view of this millennium's developments (see below) using languages of gerbes, stacks, etc.

The classification of equivalence classes of star products for *symplectic* manifolds can be obtained from there. More concretely it follows from the fact (Nest and Tsygan [NT95]) that any differentiable star product \star on a symplectic manifold M is equivalent to a star product constructed with Fedosov's algorithm. This permits to define the characteristic class of a star product as the class of Weyl curvature $H_{dR}^2(M)[[\nu]]$ (formal series in de Rham cohomology) associated to any Fedosov star-product equivalent to \star . As a consequence, one gets a parametrization of the equivalence classes of star-products on (M, ω) by elements in $H_{dR}^2(M)[[\nu]]$. That parametrization of equivalence classes of star-products has also been made

explicit by Bertelson, Cahen and Gutt [BCG97]. They are in (1–1) correspondence with formal deformations of the symplectic form,

The case of Poisson manifolds was harder to deal with (except for regular Poisson manifolds, for which all the symplectic leaves have the same dimension). Kontsevich first showed [Ko96] that what was then his “formality conjecture” implies that any Poisson manifold can be formally quantized, giving strong evidence for the conjecture to be true. A year later he came with a seminal work [Ko97] that has been extensively rewritten and developed by many, including himself (e.g. [Ko99] after Tamarkin came with an operadic approach [Hi03]). These papers contain many important notions and results, far beyond existence and classification (in (1–1) correspondence with formal deformations of a Poisson tensor).

Super-Remark. Supersymmetry became popular in the 70’s. [Incidentally a first example of the super Poincaré algebra can be found in [FH70] where the spinorial translations, an \mathbb{R}^4 Lie algebra supplementing the \mathbb{R}^4 algebra of vector space-time translations, were at some point multiplied by an anticommuting operator denoted by F , in effect producing the super Poincaré algebra. Both Wess and Zumino told me much later that they did not know of the paper.] A natural consequence of the introduction of supersymmetry was to develop, especially in the 80’s and 90’s, “supersymmetric quantum mechanics” (see e.g. an excellent review in [CKS95]). The extension of deformation quantization to supermanifolds was thus a natural thing to do. We mentioned the issue in some early papers but it is only at the end of the 90’s that a number of scientists (including Neumaier) considered it in a more precise way (see e.g. [Bo00]). So far that did not go beyond adapting Fedosov’s construction to the context of supermanifolds. [Bordemann informed me recently that at the time he put on the problem a good student, who unfortunately left science shortly after.] A nice “warm up exercise” for a good graduate student would thus be to start with treating the supersymmetric harmonic oscillator in deformation quantization.

3.2 More general context (varieties and singular spaces, Lie groupoids and algebroids, field theory, etc.)

In the past decade the geometrical context of deformation quantization (DQ) has been extended from manifolds to algebraic geometry and a variety of more general structures, real or complex, often singular in some sense. DQ became also more used in physics, unfortunately not (yet) so much in developing rigorously new examples of “autonomous quantization” (when needed without Hilbert space but with appropriate auxiliary conditions) but often, at best at the “physics level of rigor”, in the strings framework and in field theory on “noncommutative space-time” (an approach developed in particular in works by Grosse, Rivasseau, and coworkers, that can be found on arXiv). Here also these works are too numerous to be detailed, or even all mentioned, in such a short overview. Some are quoted in a

recent review [St11] (and references therein). We shall be satisfied with indicating a few directions closer to the original formulation of DQ.

Already in the 90's (see e.g. [NT95] and later works) Nest and Tsygan had extended the initial framework of DQ in various directions, in particular various kinds of index theorems (these had earlier brought to DQ Fedosov, and Connes from a different point of view, see e.g. [Co94] and references therein). With other coworkers they obtained a variety of sophisticated extensions (see e.g. a couple of the more recent [DTT09, BGNT11]), somewhat in the spirit of [De95], using gerbes, algebroid stacks, etc., and the connection with formality crucial in Kontsevich [Ko96, Ko97]. Listing all these works and their content would be a long review paper in itself, so we shall stop here.

At this stage it should be clear that index theorems and DQ are intimately related. In fact this should have been clear to us from the start. Indeed in 1963/64 I participated (together with Louis Boutet de Monvel and others) in [CS64], where my part was the multiplicative property of the analytic index of elliptic operators. But neither I until we had completed our papers [BFFLS], nor Boutet de Monvel (see his footnote in the obituary for Moshé Flato in *Gazette des Mathématiciens*, No 81, 1999) until much later (nor even Flato who attended part of the Seminar), realized that, like Mr. Jourdain speaking prose, we were then dealing with star-products of symbols! The initial index theorem was soon extended (see Atiyah's exposé in [CS64]) to manifolds with boundary. Natural (nontrivial) extensions are then to cones and manifolds with corners, and to algebraic varieties and to singular spaces.

The former gave rise to what is often called the “Melrose b -calculus” and its generalizations, see e.g. [LP05, MR11], the connection of which with DQ has not been much studied. The same can be said of the related sophisticated new approach to geometric quantization (in particular using higher structures such as gerbes) developed in past years by Mathai and coworkers (see e.g. [BMW12]).

It is important to remember that when one goes beyond the differentiable framework (see e.g. (2.3.2.2) in [St11]), the situation can change significantly. E.g. the Harrison cohomology does not vanish in general, permitting nontrivial abelian deformations, which could be of interest to quantize Nambu brackets (replacing the usual product with an abelian deformation in the defining matrices) in a more direct manner than what was done in [DFST]. The cases of complex analytic manifolds and of algebraic varieties present many intricacies, as indicated in [St11]. Specific examples of interest can be seen in [FK07] and [Fr09] (on the *closure* of minimal coadjoint orbits, a situation which resisted “conventional” treatments as mentioned in [GW08]). The treatment of the Berezin-Toeplitz quantization reviewed in [Sc10] is also relevant to geometric objects in the complex context.

3.3 Some words on highlights in two related “avatars” (quantum groups and quantized spaces)

A transition to the next section is provided by this “tachyonic” overview of what can be considered (at least from a chronological point of view) two major avatars of DQ. Many frontier works continue to appear in both. We shall here be satisfied with mentioning their existence and give a minimum of relevant references. We have mentioned before the relation with quantum groups, presented e.g. in [BGGs]. Both “avatars” show up also in recent reviews such as [St08, St11]. A natural (highly nontrivial) combination by Nest and Tsygan (to be posted soon) of all these aspects of quantization is the simultaneous quantization, in a compatible way, of all three notions involved in Hamiltonian actions of Poisson-Lie groups on Poisson manifolds.

Among the many works in the framework of noncommutative (NC) geometry we mention here only the NC manifolds of Connes and coworkers, developed in the Riemannian context. That approach can be seen as “dual” of the quantum groups algebra approach (in the same sense as, for a commutative topological algebra, its “spectrum” is the Gelfand dual). They study in particular noncommutative spheres (of low dimensions) cf. e.g. [CD03]. These are realized as “spectral triples” $(\mathcal{A}, \mathcal{H}, D)$ where \mathcal{A} is some algebra acting on a Hilbert space \mathcal{H} and D is a “Dirac operator” with compact resolvent such that $[D, a]$ extends to a bounded operator on \mathcal{H} . The idea is to generalize Riemannian geometry to the noncommutative setting.

In Section 4.2.5 we briefly indicate *how* a similar approach can be developed, *mutatis mutandi* (we [BCSV] are in the Lorentzian framework, not the Euclidean version), in the case of an hyperbolic sphere AdS, and *what* it may conjecturally be good for. The main part of next Section deals with new ideas around symmetries and their quantization. Hopefully that will explain *why* we would like to promote such a potentially revolutionary general framework,

4 Conjectural perspectives around quantized Anti de Sitter (AdS)

4.1 Deforming symmetries: Poincaré to anti de Sitter

As we mentioned in the Introduction, in 1964, shortly after Flato’s arrival in Paris and 7 years before Neumaier was born, appeared the founding paper by Gerstenhaber [Ge64] on deformation theory. It became gradually clear to many that the idea of deformation is crucial in physics, first from the symmetries point of view, but eventually also as expressed in what I call “Flato’s deformation philosophy” [Fl82].

Practically everybody concurs that two major breakthroughs revolutionized physics: relativity (special and general) and quantum mechanics. Both occurred in the first half of the twentieth century and, in spite of many advances, reconciling them is not yet achieved. We have seen that, in the second half of last century, both were interpreted as deformations of the mathematical frameworks associated with previously accepted theories. In this short paper we concentrated so far on the latter aspect, quantization. We shall now try and consider both as simultaneously as possible, from the point of view of the deformation philosophy.

Dealing with symmetries of particles, from the point of view of particle interpretation, for massless particles at least [AFFS], since we want that the momentum of particles be bounded below, among the only two possible choices in the Lie group category, the natural deformation of the Poincaré group is not the de Sitter group $SO(1, 4)$ but the anti de Sitter (AdS) group $SO(2, 3)$ (or some covering of it) and its most degenerate (lowest weight) representations. Flat Minkowski space-time is then deformed to a (negatively curved) AdS universe. That has, since the end of the seventies, given rise to many papers (in part mentioned in [St08] and references therein). These deal with what we call “singleton physics”. In particular *massless particles are composite*, not only kinematically (from the point of view of symmetries) but also dynamically for the photon (now the only truly massless particle), in a manner compatible with quantum electrodynamics à la Dyson [FF88]. Later it was shown, combining the $U(2)$ symmetry of electroweak interactions with flavor symmetry, that leptons may also be considered [Frø00] as initially massless composites of singletons, massified by interactions with 5 pairs of Higgs-like particles; such a model predicts the existence of two new mesons (“flavor analogs” to the W and Z mesons of electroweak theory), albeit with a large mass difference due to the large mass differences between the three lepton generations – unless, somewhat like for massive neutrinos, the “physical” bosons are linear combinations of those appearing in the theory. An important question is then, in such a space-time based approach, how to deal with hadrons (heavier strongly interacting particles). The very ambitious “deformation framework” that we sketch in the remainder of the paper might, as a by product, provide answers to such a question.

4.2 Some bold mathematical ideas around quantized AdS at root of unity and “affinizations”

4.2.1 qAdS symmetry at root of unity

It is a known fact among specialists that quantum groups at root of unity have special properties. In particular it has been observed in 1993 [FHT93] that the quantized AdS group at even root of unity has finite-dimensional unitary irreducible representations (UIRs). The fact was later rediscovered and somewhat extended in several papers, in particular [Sta98]. As we mention in the introduc-

tion of [BCSV], one is then tempted to consider Minkowski space-time and the Poincaré group as limits of these q AdS counterparts when $q\rho \rightarrow -0$, where $q = e^{i\theta}$ is the quantum group deformation parameter and ρ the curvature of AdS space.

A natural idea is then to try and use such UIRs as possible substitutes to the unitary symmetries of particle spectroscopy. Quantum groups, being deformations of the Hopf algebras associated with Lie groups and enveloping algebras, can be expected to behave nicely with respect to tensor product. That is probably true of the generic case but the case of even root of unity seems to be special. In particular already for what they call the finite-dimensional quantum group associated to $\mathfrak{sl}(2)$, when the quantum parameter q is a $2p^{\text{th}}$ root of unity, it has been recently shown in [KS11] that strange things may happen, showing in particular that the category of representations cannot be braided; the case of higher ranks seems hopelessly wilder³.

Nevertheless this does not mean that for physical applications these symmetries cannot be considered. On the contrary, with some luck, the situation could turn out to be better, since “nice” behavior could be restricted to a few exceptional cases for q AdS. That is a difference between the approaches of mathematicians and of physicists in many contexts: mathematicians (even when they start ‘in petto’ with specific examples) tend to study general cases, while physicists care mainly for the particular cases needed for their models or theory, which usually means low ranks and low dimensions. The idea would then be to try such (so far hypothetical) very special q AdS representations as alternatives for the (compact) “internal symmetries” (unitary groups) empirically (and successfully, see below) used for more than 60 years to classify elementary particles. The advantage of such an approach is that it would give conceptual foundation to an alternative to the empirically introduced symmetries. The price to pay is that it requires hard mathematics, which only now might become within reach.

4.2.2 “Superized” and/or affine q AdS

The above idea is however possibly too naive, in particular since one cannot at present be satisfied with simple particle spectroscopy as one was in the sixties. Since for hadrons (strongly interacting particles) half-integer spins occur, a natural idea is to complete $\mathfrak{so}(2, 3) = \mathfrak{sp}(\mathbb{R}^4)$ to its graded extension by adding (like for the Wess-Zumino super-Poincaré algebra or implicitly earlier in [FH70]) four spinorial translations whose anticommutators give $\mathfrak{sp}(\mathbb{R}^4)$. That is what happens when realizing the latter (as two coupled harmonic oscillators) with generators that are quadratic polynomials in 4 variables p_1, p_2, q_1, q_2 (see e.g. [Frø82]). The quantized versions of such superalgebras were and still are studied (see e.g. [AB97, CW12]). It would however be interesting to study further what happens for superized q AdS

³I thank M. Jimbo for drawing my attention to these facts.

at even root of unity, in particular not only for the representations themselves but also for their tensor products.

More importantly, the dynamics of the various interactions should now be part of the picture. That requires at some point to take into account singularities. A standard mathematical way to tackle such a situation, in the spirit of Hironaka⁴ is to “blow up” the singularity by introducing extra dimensions. [That is also one of the original motivations for string theory.]

Dealing with Lie groups that process is “affinization”, passing to loop groups (mappings from a closed string S^1 to the original Lie group) or affine (simple) Lie algebras, or some central extension of these. These structures can in turn be quantized and their representations studied, which is an active topic in modern mathematics with many ramifications (see e.g. recent papers such as [HJ12, JRZ10]).

4.2.3 Generalized affinizations

One may be even bolder, in (at least) two directions which are so far largely virgin territory from the mathematical point of view. A first direction is to generalize loop groups to mappings from a higher dimensional manifold or variety M to a Lie group G . For example one may take for M a K_3 surface or a Calabi-Yau (complex) 3-fold, very popular in string theory, and for G the AdS group. The mathematical problems involved appear to be very hard. Trying to “quantize” such structures would be even harder. While it could very well be hopeless to try and develop a generic theory for such structures and their representations, which is what mathematicians tend to be interested in, it might be that here also, in particular when some form of discretization is possible (which, in a different context, is what ’t Hooft very recently did [tH12]), specific cases could be manageable. That is what is of interest to physicists. It seems to be an experimental epistemological fact that often problems suggested by Nature turn out to be more seminal than problems imagined “out of the blue” by mathematicians.

4.2.4 Generalized deformations

Another general framework is to generalize the notion of deformation, beyond the theory of Gerstenhaber and even beyond multiparameter deformations, which are a natural extension that has been considered for quantum groups in the past two decades. In a couple of not so known papers [Pi97, Na98], G. Pinczon and his student F. Nadaud considered deformations in which the “deformation parameter” acts on the algebra to the left, to the right, or both ways, with interesting consequences.

But one can go even further and try to replace the scalar (complex) formal deformation parameter (an element of the group algebra over \mathbb{C} of the trivial

⁴In a way also in the spirit of e.g. Cauchy in the 19th century, with notions such as the principal value of a divergent integral, which is a distribution in the sense of L. Schwartz.

group) with something more general. Remaining in the abelian context, for multiparameter deformations, the “parameter” can be viewed as an element of the group algebra of $\mathbb{Z}/n\mathbb{Z}$, which is the center of $SU(n)$. The theory of such deformations goes along the same lines as the “G-deformations” of algebras considered by Gerstenhaber.

Going further, this time in the nonabelian direction, one can first consider quaternionic deformation parameters, which do not seem to have been really studied. There have of course been numerous attempts to develop quaternionic quantum mechanics, and books have been written on the subject, including by leading scientists. The approach belongs to what Ray Streater (see his web site) calls “lost causes of physics”. Nevertheless considering deformation quantization with deformations of algebras on the field of quaternions may have at least some mathematical interest, and developing a theory of quantum groups with such deformations may lead to interesting results.

More generally we could take as “parameter” elements of the group algebra of a finite group, e.g. the symmetric group S_n which is the Weyl group of $SU(n)$ and carries much of the information for its representation theory. That does not seem to have ever been considered; it is not even clear that the theory is still governed by some cohomology. Dealing with such deformations, starting from (the Hopf algebras associated with) the Poincaré or better AdS groups, would certainly bring interesting new mathematics. In combination with a generalization of “G-deformations” at root of unity and some “affinizations”, that might eventually provide a more fundamental approach to the symmetries and dynamics of particle physics, as we indicate in Section 4.3 below.

4.2.5 Quantized AdS space and conjectural cosmological consequences

In [BCSV] we showed how to build “quantized hyperbolic spheres”. More precisely we build (with a universal deformation formula [BGGs]) a (closed [Co94]) star product using an oscillatory integral, on a 1-dimensional extension \mathcal{R}_0 of the Heisenberg group (naturally endowed with a left invariant symplectic structure) and a Dirac operator D on the space \mathcal{H} of a regular representation of \mathcal{R}_0 . The star product endows the space \mathcal{A}^∞ of smooth vectors in \mathcal{H} with a noncommutative Fréchet algebra structure. We get in this way a noncommutative spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ à la Connes, but in a Lorentzian context, which induces on (an open \mathcal{R}_0 orbit \mathcal{M}_0 in) AdS space-time a pseudo-Riemannian deformation triple similar (except for the compactness of the resolvent) to the triples developed for quantized spheres by Connes et al. (see e.g. [CD03]). This “quantized AdS space” has an horizon which permits to consider it as a black hole (similar to the BTZ black holes [BHTZ], which exist for all AdS_n when $n \geq 3$).

For q an even root of unity, since the corresponding quantum AdS group has finite dimensional UIRs, such a quantized AdS black hole can be considered as “ q -compact” in a sense to be made precise. As we mention in [BCSV, St07], at least

in some regions of our universe, our Minkowski space-time could be, at very small distances, both deformed to anti de Sitter and quantized, to q AdS. These regions would appear as black holes which might be found at the edge of our expanding universe, a kind of “stem cells” of the initial singularity dispersed at the Big Bang. From these (that is so far mere speculation) might emerge matter, possibly first some kind of singletons that couple and become massified by interaction with dark matter and/or dark energy. Such a scheme could be responsible, at very large distances, for the observed positive cosmological constant – and might bring us a bit closer to quantizing gravity, the Holy Grail of modern physics.

4.3 Conjectural space-time origin of internal symmetries

4.3.1 On the connection between external and internal symmetries

In the mid-sixties, in view of the fundamental role of relativity in physics, a natural question was to know whether there was some connection between “external symmetries” (in particular the Poincaré group) and the empirically discovered “internal symmetries”. We co-organized a CNRS Colloque on the topic in April 1966. At that time the “internal symmetries” were mainly the $SU(3)$ of the “eight-fold way” and some subalgebras. Later color $SU(3)$ was introduced, followed with QCD to express the dynamics, ‘grand unified’ symmetries (e.g. $SU(5)$), and eventually the ‘standard model’. See e.g. a short presentation in [Ra10].

The representations of the internal symmetries gave “nice boxes” into which one could fit many newly discovered elementary particles, and predict new ones that were later found (which eventually contributed to bring to Stockholm a most influential theoretician). As we explain in [St07], the prevailing trend (in spite of our objections that “it ain’t necessarily so” due to mathematical problems) became that there is no connection except direct sum. In contradistinction with atomic or molecular spectroscopy where the (known) dynamics dictate the symmetry (e.g. a crystalline structure breaks the rotational symmetry), in this case the dynamics were eventually “invented” to fit the empirical symmetries (after the latter, e.g. $SU(3)$, changed somewhat their interpretation – but that is another story).

There is a part of self-fulfilling prophecy in the interpretations of raw experimental data that by default are made in the framework of the detailed and so far successful construct which constitutes the “standard model.” The leaders of experimental groups are of course aware of the problem (private communication from Gerard ’t Hooft), but no caveat is publicized. It could be desirable to apply to such physics recent developments in information theory such as those developed in a different context in [Re11], in order to make as much as possible model-independent the experimental data.

4.3.2 Is it necessarily so?

It eventually dawned on me that the problem of connection between symmetries, especially in the somewhat restrictive context of Lie algebras, could be a false problem. Namely, in line with our deformation philosophy, it would be quite natural that the “internal symmetries” of interacting particles *emerge* from the Poincaré symmetry of free particles via some process of (possibly generalized) deformation. We express this as follows:

Conjecture 4.1. THE DEFORMATION CONJECTURE. *Internal symmetries of elementary particles emerge from their relativistic counterparts by some form of deformation (possibly generalized, including quantization), along with “superization” and maybe a kind of “affinization”.*

Internal symmetries, especially in the modern form of the standard model which so far fits so well the spectroscopy of elementary particles, can be seen as an ultimate paradigm of quantum mechanics. Their relativistic counterparts are of geometric origin. One of the hopes of modern developments is to reconcile both and quantize gravity. That is in particular the case of the ‘strings framework’ (as say some, e.g. David Gross) and of the approach of Alain Connes and coworkers to the standard model via noncommutative geometry (which very recently found a 20% correction to its previous prediction for the mass of the Higgs boson, fitting the mass of the boson discovered at LHC).

What we are saying here is that perhaps if and when, with a lot of work and a little bit of luck, some of the mathematical avenues sketched above are successful and can be made to fit the data coming from experiments (which may have to be re-examined step by step), the sought reconciliation will in a way be a by-product. In any case the mathematical problems are worthy of attack – and can be expected to prove their worth by hitting back!

And if (in a generation or more) one avenue can be shown to fit experimental data, so much the better; one of the advantages on the experimental side is that the reconstruction of the puzzle can be achieved with the available tools, without need for more expensive ones which society can no more give us.

4.3.3 A tentative road map

The mathematical problems listed in Section 4.2 (which looks a bit like a mail order catalog) may be treated independently. But significant progress in most of them can take a long time and, as usual in research, new problems are bound to pop up. Nevertheless, since as we explained for physical applications we need only some special cases, albeit treated in much more details than a pure mathematician would be tempted to do, we shall now indicate a few directions which, with some luck, might produce within finite time the beginning of a foundation of internal symmetries on quantized relativistic symmetries, which is what we suggest here.

At first one could study some of these finite-dimensional UIRs of $q\text{AdS}$ for q an even (possibly 6^{th} since we have 3 generations) root of unity, in particular their tensor products and whether one has there something like the singletons for AdS. Then we could try and see what can be said of their affinizations and of the representations of the latter. Another direction could be to look at such deformations over the quaternions, which might have “built in” at least some of the present internal symmetries. A related direction would be to check what can be said, in the same spirit, of generalized deformations with “parameter” in the group algebra of $\mathbb{Z}/n\mathbb{Z}$ or S_n , in particular for $n = 3$. We leave further problems to the imagination of the readers who would have the patience to read the various parts of this unusual paper.

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