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Companion cluster algebras to a generalized cluster algebra

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Abstract

We study the c-vectors, g-vectors, and F-polynomials for generalized cluster algebras satisfying a normalization condition and a power condition recovering classical recursions and separation of additions formulas. We establish a relationship between the c-vectors, g-vectors, and F-polynomials of such a generalized cluster algebra and its (left- and right-) companion cluster algebras. Our main result states that the cluster variables and coefficients of the (left- and right-) companion cluster algebras can be recovered via a specialization of the F-polynomials.

1 Introduction

Cluster algebras have risen to prominence as the correct algebraic/combinatorial language for describing a certain class of recursive calculations. These recursions appear in many forms across various disciplines including Poisson geometry [GSV], combinatorics [MP], hyperbolic geometry [FG, FST, MSW], representation theory of associative algebras [CC, CK, BMRRT, R1, Q, R2], mathematical physics [EF], and quantum groups [K, GLS, KQ, BR]. In the current standard theory a product of cluster variables, one known and one unknown, is equal to a binomial in other known quantities. Recently examples have emerged in the context of hyperbolic orbifolds [CS], exact WKB analysis [IN], and quantum groups [G, BGR] that require a more general setup: these *binomial* exchange relations should be replaced by *polynomial* exchange relations.

The general study of such *generalized* cluster algebras was initiated by Chekhov and Shapiro [CS] where an analogue of the classical Laurent Phenomenon was established. Following these developments, the first author [N] studied the analogues of c-vectors, g-vectors, and F-polynomials for a class of generalized cluster algebras satisfying a *normalization condition* and a *reciprocity condition.* In that work, relationships between these c- and g-vectors with the corresponding quantities for certain *companion* cluster algebras were established. Our goal in the present paper is to extend these results to the case when the reciprocity condition is replaced by a weaker *power condition* and to clarify the corresponding relationships between F-polynomials, x-variables, and y-variables. The main message of this note, continuing from [N], is as follows: the generalized cluster algebras are as good and natural as ordinary cluster algebras. Also in this direction, analogues of the classical greedy bases from [LLZ] have been constructed for rank 2 generalized cluster algebras by the second author [R3].

In order to state our main theorem we will need to fix some notation. A cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y},B) \subset \mathcal{F}$ is defined recursively from the initial data of a seed $(\mathbf{x},\mathbf{y},B)$ where $\mathbf{y}=(y_1,\dots,y_n)$ is a collection of elements from a semifield $\mathbb{P}$, $\mathbf{x}=(x_1,\dots,x_n)$ is a collection of algebraically independent elements in a degree n purely transcendental extension $\mathcal{F}$ of $\mathbb{QP}$ (in particular, we may identify $\mathcal{F}$ with the rational function field $\mathbb{QP}(\mathbf{x})$) where $\mathbb{QP}$ is the field of fractions of the group ring $\mathbb{ZP}$, and $B=(b_{ij})$ is a skew-symmetrizable $n\times n$ matrix. A generalized cluster algebra $\mathcal{A}=\mathcal{A}(\mathbf{x},\mathbf{y},B,\mathbf{Z})\subset\mathcal{F}$ requires the additional data of a collection of *exchange polynomials* $\mathbf{Z}=(Z_1,\dots,Z_n)$ where

$$Z_i(u) = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i-1}u^{d_i-1} + z_{i,d_i}u^{d_i}$$

with each $z_{i,s}\in\mathbb{P}$ and $z_{i,0}=z_{i,d_i}=1$. Write $\mathbf{z}=(z_{i,s})$ $(1\le i\le n,\ 0\le s\le d_i)$.

Write $D=(d_i\delta_{ij})$ for the diagonal $n\times n$ matrix. Denote by $\mathbf{x}^{1/\mathbf{d}}$ the collection $(x_1^{1/d_1},\dots,x_n^{1/d_n})$ in the extension field $\mathbb{QP}(\mathbf{x}^{1/\mathbf{d}})$ of $\mathcal{F}$. Define the *left-companion cluster algebra* ${}^L\!\mathcal{A}$ of $\mathcal{A}$ to be the cluster algebra $\mathcal{A}(\mathbf{x}^{1/\mathbf{d}},\mathbf{y},DB)\subset\mathbb{QP}(\mathbf{x}^{1/\mathbf{d}})$. Write $({}^L\mathbf{x}^t,{}^L\mathbf{y}^t,{}^LB^t)$ for the seed associated to vertex $t\in\mathbb{T}_n$ in the construction of ${}^L\!\mathcal{A}$ and denote by ${}^L\mathbf{c}_j^t$, ${}^L\mathbf{g}_j^t$, and ${}^LF_j^t$ the c-vectors, g-vectors, and F-polynomials of ${}^L\!\mathcal{A}$.

Let $\mathbf{z}^{\mathrm{bin}}=(z_{i,s}^{\mathrm{bin}})$ where $z_{i,s}^{\mathrm{bin}}=\binom{d_i}{s}$. Then we write $x_i^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}}\in\mathcal{F}$ and $y_j^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}}\in\mathbb{P}$ for the variables obtained by applying equations (3.15) and (3.14) respectively using the specialized F-polynomials $F_j^t(\mathbf{y},\mathbf{z}^{\mathrm{bin}})$ in place of the generic F-polynomials $F_j^t(\mathbf{y},\mathbf{z})$. Our first main result is the following.

Theorem 1.1. *We have* $x_i^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}}=\big({}^Lx_i^t\big)^{d_i}$ *and* $y_j^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}}={}^Ly_i^t$.

Denote by $\mathbf{y}^{\mathbf{d}}$ for the collection $(y_1^{d_1},\dots,y_n^{d_n})$ in $\mathbb{P}$. Define the *right-companion cluster algebra* ${}^R\!\mathcal{A}$ of $\mathcal{A}$ to be the cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y}^{\mathbf{d}},BD)\subset\mathbb{QP}(\mathbf{x})$. Write $({}^R\mathbf{x}^t,{}^R\mathbf{y}^t,{}^RB^t)$ for the seed associated to vertex $t\in\mathbb{T}_n$ in the construction of ${}^R\!\mathcal{A}$ (see Section 2 for details).

Write $x_i^t\big|_{\mathbf{z}=\mathbf{0}}\in\mathcal{F}$ and $y_j^t\big|_{\mathbf{z}=\mathbf{0}}\in\mathbb{P}$ for the variables obtained by applying equations (3.15) and (3.14) respectively using the specialized F-polynomials $F_j^t(\mathbf{y},\mathbf{0})$ in place of the generic F-polynomials $F_j^t(\mathbf{y},\mathbf{z})$. Our second main result is the following.

Theorem 1.2. *We have* $x_i^t\big|_{\mathbf{z}=\mathbf{0}}={}^Rx_i^t$ *and* $\big(y_j^t\big|_{\mathbf{z}=\mathbf{0}}\big)^{d_j}={}^Ry_j^t$.

2 Cluster Algebras

A *semifield* is a multiplicative abelian group $(\mathbb{P}, \cdot)$ together with an auxiliary addition $\oplus : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ which is associative, commutative and satisfies the usual distributivity with the multiplication of $\mathbb{P}$. Write $\mathbb{ZP}$ for the group ring of $\mathbb{P}$. Since $\mathbb{P}$ is necessarily torsion-free (see e.g. [FZ1, Sec. 5]), $\mathbb{ZP}$ is a domain [FZ1, Sec. 2] and we write $\mathbb{QP}$ for its field of fractions. There are two main examples of semifields that will be most relevant for our purposes.

Example 2.1.

1. The *universal semifield* $\mathbb{Q}_{\mathrm{sf}}(y_1, \dots, y_n)$ is the set of rational functions in the variables $y_1, \dots, y_n$ which can be written in a subtraction-free form. Addition and multiplication in the universal semifield are the ordinary operations on rational functions. The semifield $\mathbb{Q}_{\mathrm{sf}}(y_1, \dots, y_n)$ is universal in the following sense. Each element of $\mathbb{Q}_{\mathrm{sf}}(y_1, \dots, y_n)$ can be written as a ratio of positive polynomials in $\mathbb{Z}_{\geq 0}[y_1, \dots, y_n]$ so that for any other semifield $\mathbb{P}$ there is a specialization homomorphism $\mathbb{Q}_{\mathrm{sf}}(y_1, \dots, y_n) \to \mathbb{P}$, given by $y_i \mapsto p_i$ and $1 \mapsto 1$, which respects the semifield structure for any choice of $p_1, \dots, p_n \in \mathbb{P}$.

2. The *tropical semifield* $\mathrm{Trop}(y_1, \dots, y_n)$ is the free (multiplicative) abelian group generated by $y_1, \dots, y_n$ with auxiliary addition $\oplus$ defined by
$$\prod_{j=1}^{n} y_j^{a_j} \oplus \prod_{j=1}^{n} y_j^{b_j} = \prod_{j=1}^{n} y_j^{\min(a_j, b_j)}.$$
The group ring of $\mathbb{P} = \mathrm{Trop}(y_1, \dots, y_n)$ is the Laurent polynomial ring $\mathbb{Z}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ while $\mathbb{QP} = \mathbb{Q}(y_1, \dots, y_n)$.

Fix a semifield $\mathbb{P}$ and write $\mathcal{F} = \mathbb{QP}(w_1, \dots, w_n)$ for the field of rational functions in algebraically independent variables $w_1, \dots, w_n$. A *(labeled) seed* $(\mathbf{x}, \mathbf{y}, B)$ *over* $\mathbb{P}$ consists of the following data:

- an algebraically independent collection $\mathbf{x} = (x_1, \dots, x_n)$, called a *cluster*, consisting of elements from $\mathcal{F}$ called *cluster variables* or *x-variables*;
- a collection $\mathbf{y} = (y_1, \dots, y_n)$ of elements from $\mathbb{P}$ called *coefficients* or *y-variables*;
- an $n \times n$ skew-symmetrizable matrix $B = (b_{ij})$ called the *exchange matrix*.

The main ingredient in the definition of a cluster algebra is the notion of mutation for seeds. For notational convenience we abbreviate $[b]_+ = \max(b, 0)$.

Definition 2.2. For $1 \leq k \leq n$ we define the *seed mutation in direction* k by $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ where

- the cluster $\mathbf{x}' = (x'_1, \ldots, x'_n)$ is given by $x'_i = x_i$ for $i \neq k$ and x'_k is determined using the *exchange relation*:

$$x'_k x_k = \left(\prod_{i=1}^{n} x_i^{[-b_{ik}]_+}\right) \frac{1+\hat{y}_k}{1 \oplus y_k}, \quad \hat{y}_k = y_k \prod_{i=1}^{n} x_i^{b_{ik}}; \tag{2.1}$$

- the coefficient tuple $\mathbf{y}' = (y'_1, \ldots, y'_n)$ is given by $y'_k = y_k^{-1}$ and for $j \neq k$ we set

$$y'_j = y_j y_k^{[b_{kj}]_+} \big(1 \oplus y_k\big)^{-b_{kj}}; \tag{2.2}$$

- the matrix $B' = (b'_{ij})$ is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{2.3}$$

Write $\mathbb{T}_n$ for the n-regular tree with edges labeled by the set $\{1, \ldots, n\}$ so that the n edges emanating from each vertex receive different labels. We write $t \overset{k}{—} t'$ to denote two vertices t and t' of $\mathbb{T}_n$ connected by an edge labeled by k. A *cluster pattern* Σ *over* $\mathbb{P}$ is an assignment of a seed Σ^t to each vertex $t \in \mathbb{T}_n$ such that whenever $t \overset{k}{—} t'$ we have $\mu_k \Sigma^t = \Sigma^{t'}$, that is Σ^t and $\Sigma^{t'}$ are related by the seed mutation in direction k whenever t and t' are adjoined by an edge labeled by k. Fix a choice of initial vertex t_0, we will write $\Sigma^{t_0} = (\mathbf{x}, \mathbf{y}, B)$ while for an arbitrary vertex $t \in \mathbb{T}_n$ we write $\Sigma^t = (\mathbf{x}^t, \mathbf{y}^t, B^t)$ where

$$\mathbf{x}^t = (x_1^t, \ldots, x_n^t), \qquad \mathbf{y}^t = (y_1^t, \ldots, y_n^t), \qquad B^t = (b_{ij}^t).$$

Note that every seed Σ^t for $t \in \mathbb{T}_n$ is uniquely determined once we have specified Σ^{t_0}. Moreover, it is important to note that the exchange matrices B^t are independent of the initial choice of $\mathbf{x}$ and $\mathbf{y}$.

Definition 2.3. The *cluster algebra* $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all cluster variables from seeds appearing in the cluster pattern Σ, more precisely

$$\mathcal{A}(\mathbf{x}, \mathbf{y}, B) = \mathbb{ZP}[x_i^t : t \in \mathbb{T}_n, 1 \leq i \leq n] \subset \mathcal{F}.$$

A priori the most one can say about these constructions is that the cluster variables x_i^t admit a description as subtraction-free rational expressions in the cluster variables of $\mathbf{x}$ with coefficients in $\mathbb{ZP}$ and that the coefficients y_j^t admit a description as subtraction-free rational expressions in $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$. More precisely, to see this claim for x_i^t we may, for each initial seed $(\mathbf{x}, \mathbf{y}, B)$,

- replace the x- and y-variables by formal indeterminants (which by abuse of notation we denote by the same symbols);

- replace the semifield $\mathbb{P}$ by the tropical semifield $\mathrm{Trop}(\mathbf{y})$;
- replace $\mathcal{F}$ by $\mathbb{Q}(\mathbf{x},\mathbf{y})$ and opt to perform all calculations here.

Since no subtraction occurs in the recursions (2.1), we obtain in this way *X-functions* $X_i^t \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{x},\mathbf{y})$. Alternatively performing the y-mutations (2.2) inside $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ we obtain *Y-functions* $Y_j^t \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$. By the universality of the semifield $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ we may recover the original coefficient y_j^t by the specialization $Y_j^t\big|_{\mathbb{P}}$. Taking this specialization where $\mathbb{P} = \mathrm{Trop}(\mathbf{y})$ we obtain monomials $Y_j^t\big|_{\mathrm{Trop}(\mathbf{y})} = \prod_{i=1}^n y_i^{c_{ij}^t}$ where we write C^t for the resulting matrix whose columns $\mathbf{c}_j^t \in \mathbb{Z}^n$ are called *c-vectors*. Note that the c-vectors only depend on the initial exchange matrix B and not on the choice of initial cluster $\mathbf{x}$.

Proposition 2.4. *[FZ4, Eq. 5.9] The c-vectors satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$c_{ij}^{t'} = \begin{cases} -c_{ik}^t & \text{if } j = k; \\ c_{ij}^t + c_{ik}^t[b_{kj}^t]_+ + [-c_{ik}^t]_+ b_{kj}^t & \text{if } j \neq k. \end{cases} \tag{2.4}$$

Obtaining the cluster variable x_i^t from X_i^t is more interesting and will be discussed further below. As a first step toward this goal, we note that the cluster algebra $\mathcal{A}$ admits the following remarkable "Laurent Phenomenon".

Theorem 2.5. *[FZ1, Th. 3.1] Fix an initial seed $(\mathbf{x},\mathbf{y},B)$ over a semifield $\mathbb{P}$. For any vertex $t \in \mathbb{T}_n$ each cluster variable x_i^t can be expressed as a Laurent polynomial in $\mathbf{x}$ with coefficients in $\mathbb{Z}\mathbb{P}$.*

For a seed $(\mathbf{x},\mathbf{y},B)$ over $\mathbb{P} = \mathrm{Trop}(\mathbf{y})$ we may apply Theorem 2.5 to write each X-function as an element of $\mathbb{Z}[\mathbf{x}^{\pm 1},\mathbf{y}^{\pm 1}]$. Moreover, y-variables never appear in the denominators of the X-functions.

Proposition 2.6. *[FZ4, Prop. 3.6] Each X-function X_i^t is contained in $\mathbb{Z}[\mathbf{x}^{\pm 1},\mathbf{y}]$.*

In fact, the X-functions are homogeneous with respect to a certain $\mathbb{Z}^n$-grading on $\mathbb{Z}[\mathbf{x}^{\pm 1},\mathbf{y}]$. Write $\mathbf{b}_j \in \mathbb{Z}^n$ for the j^{th} column of B.

Proposition 2.7. *[FZ4, Prop. 6.1, Prop. 6.6] Under the $\mathbb{Z}^n$-grading*

$$\deg(x_i) = \mathbf{e}_i \qquad \textit{and} \qquad \deg(y_j) = -\mathbf{b}_j,$$

each X-function is homogeneous and we write $\deg\left(X_j^t\right) = \mathbf{g}_j^t = \sum_{i=1}^n g_{ij}^t \mathbf{e}_i$. *Moreover, these g-vectors satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$g_{ij}^{t'} = \begin{cases} g_{ij}^t & \text{if } j \neq k; \\ -g_{ik}^t + \sum_{\ell=1}^n g_{i\ell}^t[-b_{\ell k}^t]_+ - \sum_{\ell=1}^n b_{i\ell}^t[-c_{\ell k}^t]_+ & \text{if } j = k. \end{cases} \tag{2.5}$$

Following Proposition 2.6 we may define *F-polynomials* $F_i^t(\mathbf{y}) \in \mathbb{Z}[\mathbf{y}]$ via the specialization $F_i^t(\mathbf{y}) = X_i^t(\mathbf{1}, \mathbf{y})$, i.e. by setting all initial cluster variables x_j to 1. The F-polynomials satisfy a recurrence relation analogous to (2.1).

Proposition 2.8. *[FZ4, Prop. 5.1] The F-polynomials satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$F_j^{t'} = \begin{cases} F_j^t & \text{if } j \neq k; \\ (F_k^t)^{-1} \left(\prod_{i=1}^n y_i^{[-c_{ik}^t]_+} (F_i^t)^{[-b_{ik}^t]_+} \right) \left(1 + \prod_{i=1}^n y_i^{c_{ik}^t} (F_i^t)^{b_{ik}^t} \right) & \text{if } j = k. \end{cases} \tag{2.6}$$

Notice that each F-polynomial admits an expression as a subtraction-free rational expression and thus may be considered as an element of $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$, in particular the specialization $F_i^t\big|_{\mathbb{P}}$ makes sense for any semifield $\mathbb{P}$. With this we may obtain a description of the y-variables in terms of the c-vectors and the specializations of the F-polynomials.

Theorem 2.9. *[FZ4, Prop. 3.13] Fix an initial seed $(\mathbf{x}, \mathbf{y}, B)$ over a semifield $\mathbb{P}$. For any vertex $t \in \mathbb{T}_n$ each coefficient y_j^t of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ can be computed as*

$$y_j^t = \left(\prod_{i=1}^n y_i^{c_{ij}^t} \right) \prod_{i=1}^n F_i^t\big|_{\mathbb{P}}(\mathbf{y})^{b_{ij}^t}.$$

Finally, we obtain a "separation of additions" formula for the cluster variables x_i^t in terms of the g-vectors and the F-polynomials.

Theorem 2.10. *[FZ4, Cor. 6.3] Fix an initial seed $(\mathbf{x}, \mathbf{y}, B)$ over a semifield $\mathbb{P}$. For any vertex $t \in \mathbb{T}_n$ each cluster variable x_j^t of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ can be computed as*

$$x_j^t = \left(\prod_{i=1}^n x_i^{g_{ij}^t} \right) \frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y})}.$$

3 Generalized Cluster Algebras

Let $(\mathbf{x}, \mathbf{y}, B)$ be a seed over the semifield $\mathbb{P}$. Fix a collection $\mathbf{Z} = (Z_1, \ldots, Z_n)$ of positive degree *exchange polynomials*

$$Z_i(u) = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i-1}u^{d_i-1} + z_{i,d_i}u^{d_i} \in \mathbb{ZP}[u]$$

such that $z_{i,s} \in \mathbb{P}$ for $0 \le s \le d_i$ and $z_{i,0} = z_{i,d_i} = 1$. It will often be convenient to write $\mathbf{z} = (z_{i,s})$ with $1 \le i \le n$ and $0 \le s \le d_i$ for the coefficients of the polynomials Z_i. Write $\overline{Z_i}(u) = u^{d_i} Z_i(u^{-1})$ for the exchange polynomial with coefficients reversed. Together we call $\Sigma = (\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ a *generalized seed over* $\mathbb{P}$. The additional data of the polynomials $\mathbf{Z}$ allows to generalize the notion of seed mutation in such a way that all nice properties and constructions related to cluster algebras in section 2 carry over to the new setting.

Definition 3.1. For $1 \leq k \leq n$ we define the *generalized seed mutation in direction* k by $\mu_k(\mathbf{x}, \mathbf{y}, B, \mathbf{Z}) = (\mathbf{x}', \mathbf{y}', B', \mathbf{Z}')$ where

- the cluster $\mathbf{x}' = (x_1', \ldots, x_n')$ is given by $x_i' = x_i$ for $i \neq k$ and x_k' is determined using the *exchange relation*:

$$x_k' x_k = \left(\prod_{i=1}^n x_i^{[-b_{ik}]_+} \right)^{d_k} \frac{Z_k\big(\hat{y}_k\big)}{Z_k\big|_{\mathbb{P}}(y_k)}, \quad \hat{y}_k = y_k \prod_{i=1}^n x_i^{b_{ik}}; \tag{3.1}$$

- the coefficient tuple $\mathbf{y}' = (y_1', \ldots, y_n')$ is given by $y_k' = y_k^{-1}$ and for $j \neq k$ we set

$$y_j' = y_j \big(y_k^{d_k}\big)^{[b_{kj}]_+} Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}; \tag{3.2}$$

- the matrix $B' = (b_{ij}')$ is given by

$$b_{ij}' = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ d_k b_{kj} + b_{ik} d_k [-b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{3.3}$$

- the exchange polynomials $\mathbf{Z}' = (Z_1', \ldots, Z_n')$ are given by $Z_i' = Z_i$ for $i \neq k$ and $Z_k' = \overline{Z}_k$, writing this relation purely in terms of coefficients gives $z_{i,s}' = z_{i,s}$ for $i \neq k$ and $z_{k,s}' = z_{k,d_k-s}$.

One may easily check that the $\hat{y}$-variables mutate in the same way as the y-variables, namely $\hat{y}_k' = \hat{y}_k^{-1}$ and for $j \neq k$ we have

$$\hat{y}_j' = \hat{y}_j \big(\hat{y}_k^{d_k}\big)^{[b_{kj}]_+} Z_k(\hat{y}_k)^{-b_{kj}}. \tag{3.4}$$

As a first indication that this definition is correct we verify that $\mu_k^2 \Sigma = \Sigma$.

Proposition 3.2. *The generalized seed mutation μ_k is involutive.*

Proof. Consider the generalized seed mutations

$$(\mathbf{x}', \mathbf{y}', B', \mathbf{Z}') = \mu_k(\mathbf{x}, \mathbf{y}, B, \mathbf{Z}) \quad \text{and} \quad (\mathbf{x}'', \mathbf{y}'', B'', \mathbf{Z}'') = \mu_k(\mathbf{x}', \mathbf{y}', B', \mathbf{Z}').$$

To begin note that $x_i'' = x_i' = x_i$ for $i \neq k$ and $\big(\hat{y}_k'\big)^{-1} = \hat{y}_k$. Then x_k'' is given by

$$\begin{aligned} x_k'' &= \frac{1}{x_k'} \left(\prod_{i=1}^n (x_i')^{[-b_{ik}']_+} \right)^{d_k} \frac{\overline{Z}_k\big(\hat{y}_k'\big)}{\overline{Z}_k\big|_{\mathbb{P}}(y_k')} = \frac{1}{x_k'} \left(\prod_{i=1}^n x_i^{[b_{ik}]_+} \right)^{d_k} \frac{\hat{y}_k^{-d_k} Z_k\big(\hat{y}_k\big)}{y_k^{-d_k} Z_k\big|_{\mathbb{P}}(y_k)} \\ &= \frac{1}{x_k'} \left(\prod_{i=1}^n x_i^{[b_{ik}]_+ - b_{ik}} \right)^{d_k} \frac{Z_k\big(\hat{y}_k\big)}{Z_k\big|_{\mathbb{P}}(y_k)} = \frac{1}{x_k'} \left(\prod_{i=1}^n x_i^{[-b_{ik}]_+} \right)^{d_k} \frac{Z_k\big(\hat{y}_k\big)}{Z_k\big|_{\mathbb{P}}(y_k)} = x_k. \end{aligned}$$

Also $y''_k = (y'_k)^{-1} = y_k$, while for $j \neq k$ we have

$$\begin{aligned} y''_j &= y'_j\big((y'_k)^{d_k}\big)^{[b'_{kj}]_+}\overline{Z}_k\big|_{\mathbb{P}}(y'_k)^{-b'_{kj}} \\ &= y_j\big(y_k^{d_k}\big)^{[b_{kj}]_+}Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}\big(y_k^{d_k}\big)^{-[-b_{kj}]_+}\Big(y_k^{-d_k}Z_k\big|_{\mathbb{P}}(y_k)\Big)^{b_{kj}} = y_j. \end{aligned}$$

To see that the matrix mutation is involutive notice that we may apply the classical matrix mutation (2.3) to obtain exchange matrices $(DB)'$ and $(BD)'$ where $D = (d_i\delta_{ij})$. Then it is immediate from (3.3) that we have $DB' = (DB)'$ and $B'D = (BD)'$, the involutivity of matrix mutation (3.3) follows. Finally the equality $\mathbf{Z}'' = \mathbf{Z}$ is immediate from the definitions. □

The generalized seeds and their mutations we have defined here are a specialization of the setup in [CS]. There a generalized seed over $\mathbb{P}$ is a triple $(\mathbf{x}, \mathbf{p}, B)$ where $\mathbf{x}$ is a cluster, B is an exchange matrix, and $\mathbf{p} = (p_{i,s})$, where $1 \leq i \leq n$ and $0 \leq s \leq d_i$, is a collection of elements of $\mathbb{P}$. The mutation $\mu_k(\mathbf{x}, \mathbf{p}, B) = (\mathbf{x}', \mathbf{p}', B')$ is given by replacing (3.1) with

$$x'_k x_k = \left(\prod_{i=1}^n x_i^{[-b_{ik}]_+}\right)^{d_k}\sum_{s=0}^{d_k} p_{k,s}w_k^s, \quad w_k = \prod_{i=1}^n x_i^{b_{ik}} \tag{3.5}$$

and by replacing (3.2) with

$$p'_{k,s} = p_{k,d_k-s} \quad \text{and} \quad \frac{p'_{j,s}}{p'_{j,0}} = \frac{p_{j,s}}{p_{j,0}}\left(\frac{p_{k,d_k}}{p_{k,0}}\right)^{s[b_{kj}]_+} p_{k,0}^{sb_{kj}}.$$

Our generalized seed mutations can be related to the more general setting of [CS] by defining

$$p_{i,s} = \frac{z_{i,s}y_i^s}{Z_i\big|_{\mathbb{P}}(y_i)} \tag{3.6}$$

where we note the identities

$$\bigoplus_{s=0}^{d_i} p_{i,s} = 1 \quad \text{and} \quad \frac{p_{i,d_i}}{p_{i,0}} = y_i^{d_i}.$$

Proposition 3.3. *Generalized seeds of the form $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ are in bijection with generalized seeds of the form $(\mathbf{x}, \mathbf{p}, B)$ satisfying*

1. *(normalization condition)* $\bigoplus_{s=0}^{d_i} p_{i,s} = 1$;
2. *(power condition)* $\frac{p_{i,d_i}}{p_{i,0}} = y_i^{d_i}$ *for some* $y_i \in \mathbb{P}$.

Moreover, this bijection is compatible with mutations.

Remark 3.4. Such y_i as in (2) is unique since $\mathbb{P}$ is torsion-free, i.e. if $(y'_i)^{d_i} = y_i^{d_i}$ then $\left(\frac{y'_i}{y_i}\right)^{d_i} = 1$ and so $\frac{y'_i}{y_i} = 1$.

Proof. For a generalized seed $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ define $p_{i,s}$ as in (3.6). Write $(\mathbf{x}', \mathbf{y}', B', \mathbf{Z}') = \mu_k(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ and again use (3.6) to define $p'_{i,s}$ in terms of this seed. Then we have

$$p'_{k,s} = \frac{z'_{k,s}(y'_k)^s}{\overline{Z}_k\big|_{\mathbb{P}}(y'_k)} = \frac{z_{k,d_k-s}y_k^{-s}}{y_k^{-d_k}Z_k\big|_{\mathbb{P}}(y_k)} = \frac{z_{k,d_k-s}y_k^{d_k-s}}{Z_k\big|_{\mathbb{P}}(y_k)} = p_{k,d_k-s}$$

while for $j \neq k$ we have

$$\frac{p'_{j,s}}{p'_{j,0}} = \frac{z'_{j,s}(y'_j)^s}{\overline{Z}_j\big|_{\mathbb{P}}(y'_j)}\frac{\overline{Z}_j\big|_{\mathbb{P}}(y'_j)}{z'_{j,0}} = z_{j,s}\left(y_j\left(y_k^{d_k}\right)^{[b_{kj}]_+}Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}\right)^s = \frac{p_{j,s}}{p_{j,0}}\left(\frac{p_{k,d_k}}{p_{k,0}}\right)^{s[b_{kj}]_+}p_{k,0}^{sb_{kj}}$$

as desired.

Conversely, let $(\mathbf{x}, \mathbf{p}, B)$ be a generalized seed satisfying (1) and (2) where we define y_i using (2). Set $z_{i,s} = y_i^{-s}\frac{p_{i,s}}{p_{i,0}}$. Notice that the definitions immediately imply $z_{i,0} = z_{i,d_i} = 1$. Since $p_{i,s} = z_{i,s}y_i^s p_{i,0}$, by the normalization condition we have $p_{i,0}^{-1} = Z_i|_{\mathbb{P}}(y_i)$ where we write $Z_i = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i-1}u^{d_i-1} + z_{i,d_i}u^{d_i}$. Write $(\mathbf{x}', \mathbf{p}', B') = \mu_k(\mathbf{x}, \mathbf{p}, B)$ so that we may define y'_i and $z'_{i,s}$ as above using this generalized seed. Then we have

$$(y'_k)^{d_k} = \frac{p'_{k,d_k}}{p'_{k,0}} = \frac{p_{k,0}}{p_{k,d_k}} = y_k^{-d_k}$$

while for $j \neq k$ we have

$$(y'_j)^{d_j} = \frac{p'_{j,d_j}}{p'_{j,0}} = \frac{p_{j,d_j}}{p_{j,0}}\left(\frac{p_{k,d_k}}{p_{k,0}}\right)^{d_j[b_{kj}]_+}p_{k,0}^{d_jb_{kj}} = \left(y_j\left(y_k^{d_k}\right)^{[b_{kj}]_+}Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}\right)^{d_j},$$

so the coefficients mutate as desired. Similarly we have

$$z'_{k,s} = (y'_k)^{-s}\frac{p'_{k,s}}{p'_{k,0}} = y_k^s\frac{p_{k,d_k-s}}{p_{k,d_k}} = y_k^s\frac{p_{k,d_k-s}}{p_{k,0}}\frac{p_{k,0}}{p_{k,d_k}} = y_k^s y_k^{d_k-s}z_{k,d_k-s}y_k^{-d_k} = z_{k,d_k-s}$$

and for $j \neq k$ we have

$$z'_{j,s} = (y'_j)^{-s}\frac{p'_{j,s}}{p'_{j,0}} = \left(y_j\left(y_k^{d_k}\right)^{[b_{kj}]_+}Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}\right)^{-s}\frac{p_{j,s}}{p_{j,0}}\left(\frac{p_{k,d_k}}{p_{k,0}}\right)^{s[b_{kj}]_+}p_{k,0}^{sb_{kj}} = z_{j,s}$$

as desired. □

A *generalized cluster pattern* Σ *over* $\mathbb{P}$ is in assignment of a generalized seed Σ^t to each vertex $t \in \mathbb{T}_n$ such that whenever $t \overset{k}{\text{—}} t'$ we have $\mu_k \Sigma^t = \Sigma^{t'}$. As for cluster algebras, the entire generalized cluster pattern Σ is uniquely determined from any choice of initial seed $\Sigma^{t_0} = (\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$. We maintain the notation $\Sigma^t = (\mathbf{x}^t, \mathbf{y}^t, B^t, \mathbf{Z}^t)$ from above where we write $\mathbf{Z}^t = (Z_1^t, \dots, Z_n^t)$.

Definition 3.5. The *generalized cluster algebra* $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ is the $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by all cluster variables from seeds appearing in the generalized cluster pattern Σ, more precisely

$$\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z}) = \mathbb{Z}\mathbb{P}[x_i^t : t \in \mathbb{T}_n, 1 \le i \le n] \subset \mathcal{F}.$$

The main feature of cluster algebras to which one might attribute their ubiquity is the Laurent Phenomenon, a first indication that generalized cluster algebras will find themselves as useful is the following consequence of Proposition 3.3 and [CS, Th. 2.5].

Corollary 3.6. *Fix an initial generalized seed* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *over a semifield* $\mathbb{P}$. *For any vertex* $t \in \mathbb{T}_n$ *each cluster variable* x_i^t *can be expressed as a Laurent polynomial of* $\mathbf{x}$ *with coefficients in* $\mathbb{Z}\mathbb{P}$.

Example 3.7. Consider the rank 2 generalized seed $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ over $\mathbb{P}$ where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{Z} = (Z_1, Z_2)$ where $Z_1(u) = 1 + z_1 u + z_2 u^2 + u^3$ and $Z_2(u) = 1 + u$. In this case we have $\hat{y}_1 = y_1 x_2$ and $\hat{y}_2 = y_2 x_1^{-1}$. Write $\Sigma(1) = (\mathbf{x}(1), \mathbf{y}(1), B(1), \mathbf{Z}(1))$ for the initial generalized seed $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ and define seeds $\Sigma(t)$ for $t = 2, \dots, 9$ inductively via the alternating mutation sequence below:

(3.7)

$$\Sigma(1) \overset{\mu_1}{\longleftrightarrow} \Sigma(2) \overset{\mu_2}{\longleftrightarrow} \Sigma(3) \overset{\mu_1}{\longleftrightarrow} \Sigma(4) \overset{\mu_2}{\longleftrightarrow} \Sigma(5) \overset{\mu_1}{\longleftrightarrow} \Sigma(6) \overset{\mu_2}{\longleftrightarrow} \Sigma(7) \overset{\mu_1}{\longleftrightarrow} \Sigma(8) \overset{\mu_2}{\longleftrightarrow} \Sigma(9).$$

Then the exchange matrices and exchange polynomials of these generalized seeds are given by

$$B(t) = (-1)^{t+1} B, \qquad Z_2(t) = Z_2, \qquad \text{and} \qquad Z_1(t) = \begin{cases} Z_1 & \text{if } t \text{ is odd;} \\ \overline{Z}_1 & \text{if } t \text{ is even.} \end{cases}$$

The resulting cluster variables and coefficients are presented in Table 1.

Following the same formal procedure as in section 2, we may define *X-functions* $X_i^t \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and *Y-functions* $Y_j^t \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{y}, \mathbf{z})$ by computing x_i^t and y_i^t, respectively, in the field $\mathbb{Q}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ where $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ represent collections of formal indeterminants. Using that $z_{i,0} = z_{i,d_i} = 1$, the specialization of the Y-functions in the tropical semifield $\mathbb{P} = \mathrm{Trop}(\mathbf{y}, \mathbf{z})$ again produces monomials $Y_j^t\big|_{\mathrm{Trop}(\mathbf{y},\mathbf{z})} = \prod_{i=1}^{n} y_i^{c_{ij}^t}$ where we write C^t for the resulting matrix whose columns $\mathbf{c}_j^t \in \mathbb{Z}^n$ we continue to call *c-vectors*.

$$\begin{cases} x_1(1) = x_1 \\ x_2(1) = x_2 \end{cases} \qquad \begin{cases} y_1(1) = y_1 \\ y_2(1) = y_2 \end{cases}$$

$$\begin{cases} x_1(2) = x_1^{-1} \frac{1+z_1\hat{y}_1+z_2\hat{y}_1^2+\hat{y}_1^3}{1\oplus z_1y_1\oplus z_2y_1^2\oplus y_1^3} \\ x_2(2) = x_2 \end{cases} \qquad \begin{cases} y_1(2) = y_1^{-1} \\ y_2(2) = y_2(1\oplus z_1y_1\oplus z_2y_1^2\oplus y_1^3) \end{cases}$$

$$\begin{cases} x_1(3) = x_1^{-1} \frac{1+z_1\hat{y}_1+z_2\hat{y}_1^2+\hat{y}_1^3}{1\oplus z_1y_1\oplus z_2y_1^2\oplus y_1^3} \\ x_2(3) = x_2^{-1} \frac{1+\hat{y}_2+z_1\hat{y}_1\hat{y}_2+z_2\hat{y}_1^2\hat{y}_2+\hat{y}_1^3\hat{y}_2}{1\oplus y_2\oplus z_1y_1y_2\oplus z_2y_1^2y_2\oplus y_1^3y_2} \end{cases} \qquad \begin{cases} y_1(3) = y_1^{-1}(1\oplus y_2\oplus z_1y_1y_2\oplus z_2y_1^2y_2\oplus y_1^3y_2) \\ y_2(3) = y_2^{-1}(1\oplus z_1y_1\oplus z_2y_1^2\oplus y_1^3)^{-1} \end{cases}$$

$$\begin{cases} x_1(4) = x_1x_2^{-3} \frac{\substack{1+3\hat{y}_2+3\hat{y}_2^2+\hat{y}_2^3+2z_1\hat{y}_1\hat{y}_2+4z_1\hat{y}_1\hat{y}_2^2+2z_1\hat{y}_1\hat{y}_2^3+z_2\hat{y}_1^2\hat{y}_2+z_1^2\hat{y}_1^2\hat{y}_2^2+3z_2\hat{y}_1^2\hat{y}_2^2+z_1^2\hat{y}_1^2\hat{y}_2^3+2z_2\hat{y}_1^2\hat{y}_2^3 \\ +3\hat{y}_1^3\hat{y}_2^2+z_1z_2\hat{y}_1^3\hat{y}_2^2+2\hat{y}_1^3\hat{y}_2^3+2z_1z_2\hat{y}_1^3\hat{y}_2^3+z_1\hat{y}_1^4\hat{y}_2^2+2z_1\hat{y}_1^4\hat{y}_2^3+z_2^2\hat{y}_1^4\hat{y}_2^3+2z_2\hat{y}_1^5\hat{y}_2^3+\hat{y}_1^6\hat{y}_2^3}}{\substack{1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus 2z_1y_1y_2\oplus 4z_1y_1y_2^2\oplus 2z_1y_1y_2^3\oplus z_2y_1^2y_2\oplus z_1^2y_1^2y_2^2\oplus 3z_2y_1^2y_2^2\oplus z_1^2y_1^2y_2^3\oplus 2z_2y_1^2y_2^3 \\ \oplus 3y_1^3y_2^2\oplus z_1z_2y_1^3y_2^2\oplus 2y_1^3y_2^3\oplus 2z_1z_2y_1^3y_2^3\oplus z_1y_1^4y_2^2\oplus 2z_1y_1^4y_2^3\oplus z_2^2y_1^4y_2^3\oplus 2z_2y_1^5y_2^3\oplus y_1^6y_2^3}} \\ x_2(4) = x_2^{-1} \frac{1+\hat{y}_2+z_1\hat{y}_1\hat{y}_2+z_2\hat{y}_1^2\hat{y}_2+\hat{y}_1^3\hat{y}_2}{1\oplus y_2\oplus z_1y_1y_2\oplus z_2y_1^2y_2\oplus y_1^3y_2} \end{cases}$$

$$\begin{cases} y_1(4) = y_1(1\oplus y_2\oplus z_1y_1y_2\oplus z_2y_1^2y_2\oplus y_1^3y_2)^{-1} \\ y_2(4) = y_1^{-3}y_2^{-1}(1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus 2z_1y_1y_2\oplus 4z_1y_1y_2^2\oplus 2z_1y_1y_2^3 \\ \qquad \oplus z_2y_1^2y_2\oplus z_1^2y_1^2y_2^2\oplus 3z_2y_1^2y_2^2\oplus z_1^2y_1^2y_2^3\oplus 2z_2y_1^2y_2^3 \\ \qquad \oplus z_1z_2y_1^3y_2^2\oplus 2z_1z_2y_1^3y_2^3\oplus 3y_1^3y_2^2\oplus 2y_1^3y_2^3 \\ \qquad \oplus z_1y_1^4y_2^2\oplus 2z_1y_1^4y_2^3\oplus z_2^2y_1^4y_2^3\oplus 2z_2y_1^5y_2^3\oplus y_1^6y_2^3) \end{cases}$$

$$\begin{cases} x_1(5) = x_1x_2^{-3} \frac{\substack{1+3\hat{y}_2+3\hat{y}_2^2+\hat{y}_2^3+2z_1\hat{y}_1\hat{y}_2+4z_1\hat{y}_1\hat{y}_2^2+2z_1\hat{y}_1\hat{y}_2^3+z_2\hat{y}_1^2\hat{y}_2+z_1^2\hat{y}_1^2\hat{y}_2^2+3z_2\hat{y}_1^2\hat{y}_2^2+z_1^2\hat{y}_1^2\hat{y}_2^3+2z_2\hat{y}_1^2\hat{y}_2^3 \\ +3\hat{y}_1^3\hat{y}_2^2+z_1z_2\hat{y}_1^3\hat{y}_2^2+2\hat{y}_1^3\hat{y}_2^3+2z_1z_2\hat{y}_1^3\hat{y}_2^3+z_1\hat{y}_1^4\hat{y}_2^2+2z_1\hat{y}_1^4\hat{y}_2^3+z_2^2\hat{y}_1^4\hat{y}_2^3+2z_2\hat{y}_1^5\hat{y}_2^3+\hat{y}_1^6\hat{y}_2^3}}{\substack{1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus 2z_1y_1y_2\oplus 4z_1y_1y_2^2\oplus 2z_1y_1y_2^3\oplus z_2y_1^2y_2\oplus z_1^2y_1^2y_2^2\oplus 3z_2y_1^2y_2^2\oplus z_1^2y_1^2y_2^3\oplus 2z_2y_1^2y_2^3 \\ \oplus 3y_1^3y_2^2\oplus z_1z_2y_1^3y_2^2\oplus 2y_1^3y_2^3\oplus 2z_1z_2y_1^3y_2^3\oplus z_1y_1^4y_2^2\oplus 2z_1y_1^4y_2^3\oplus z_2^2y_1^4y_2^3\oplus 2z_2y_1^5y_2^3\oplus y_1^6y_2^3}} \\ x_2(5) = x_1x_2^{-2} \frac{1+2\hat{y}_2+\hat{y}_2^2+z_1\hat{y}_1\hat{y}_2+z_1\hat{y}_1\hat{y}_2^2+z_2\hat{y}_1^2\hat{y}_2^2+\hat{y}_1^3\hat{y}_2^2}{1\oplus 2y_2\oplus y_2^2\oplus z_1y_1y_2\oplus z_1y_1y_2^2\oplus z_2y_1^2y_2^2\oplus y_1^3y_2^2} \end{cases}$$

$$\begin{cases} y_1(5) = y_1^{-2}y_2^{-1}(1\oplus 2y_2\oplus y_2^2\oplus z_1y_1y_2\oplus z_1y_1y_2^2\oplus z_2y_1^2y_2^2\oplus y_1^3y_2^2) \\ y_2(5) = y_1^3y_2(1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus 2z_1y_1y_2\oplus 4z_1y_1y_2^2\oplus 2z_1y_1y_2^3 \\ \qquad \oplus z_2y_1^2y_2\oplus z_1^2y_1^2y_2^2\oplus 3z_2y_1^2y_2^2\oplus z_1^2y_1^2y_2^3\oplus 2z_2y_1^2y_2^3 \\ \qquad \oplus 3y_1^3y_2^2\oplus z_1z_2y_1^3y_2^2\oplus 2y_1^3y_2^3\oplus 2z_1z_2y_1^3y_2^3 \\ \qquad \oplus z_1y_1^4y_2^2\oplus 2z_1y_1^4y_2^3\oplus z_2^2y_1^4y_2^3\oplus 2z_2y_1^5y_2^3\oplus y_1^6y_2^3)^{-1} \end{cases}$$

$$\begin{cases} x_1(6) = x_1^2x_2^{-3} \frac{1+3\hat{y}_2+3\hat{y}_2^2+\hat{y}_2^3+z_1\hat{y}_1\hat{y}_2+2z_1\hat{y}_1\hat{y}_2^2+z_1\hat{y}_1\hat{y}_2^3+z_2\hat{y}_1^2\hat{y}_2^2+z_2\hat{y}_1^2\hat{y}_2^3+\hat{y}_1^3\hat{y}_2^3}{1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus z_1y_1y_2\oplus 2z_1y_1y_2^2\oplus z_1y_1y_2^3\oplus z_2y_1^2y_2^2\oplus z_2y_1^2y_2^3\oplus y_1^3y_2^3} \\ x_2(6) = x_1x_2^{-2} \frac{1+2\hat{y}_2+\hat{y}_2^2+z_1\hat{y}_1\hat{y}_2+z_1\hat{y}_1\hat{y}_2^2+z_2\hat{y}_1^2\hat{y}_2^2+\hat{y}_1^3\hat{y}_2^2}{1\oplus 2y_2\oplus y_2^2\oplus z_1y_1y_2\oplus z_1y_1y_2^2\oplus z_2y_1^2y_2^2\oplus y_1^3y_2^2} \end{cases}$$

$$\begin{cases} y_1(6) = y_1^2y_2(1\oplus 2y_2\oplus y_2^2\oplus z_1y_1y_2\oplus z_1y_1y_2^2\oplus z_2y_1^2y_2^2\oplus y_1^3y_2^2)^{-1} \\ y_2(6) = y_1^{-3}y_2^{-2}(1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus z_1y_1y_2\oplus 2z_1y_1y_2^2\oplus z_1y_1y_2^3 \\ \qquad \oplus z_2y_1^2y_2^2\oplus z_2y_1^2y_2^3\oplus y_1^3y_2^3) \end{cases}$$

$$\begin{cases} x_1(7) = x_1^2x_2^{-3} \frac{1+3\hat{y}_2+3\hat{y}_2^2+\hat{y}_2^3+z_1\hat{y}_1\hat{y}_2+2z_1\hat{y}_1\hat{y}_2^2+z_1\hat{y}_1\hat{y}_2^3+z_2\hat{y}_1^2\hat{y}_2^2+z_2\hat{y}_1^2\hat{y}_2^3+\hat{y}_1^3\hat{y}_2^3}{1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus z_1y_1y_2\oplus 2z_1y_1y_2^2\oplus z_1y_1y_2^3\oplus z_2y_1^2y_2^2\oplus z_2y_1^2y_2^3\oplus y_1^3y_2^3} \\ x_2(7) = x_1x_2^{-1} \frac{1+\hat{y}_2}{1\oplus y_2} \end{cases}$$

$$\begin{cases} y_1(7) = y_1^{-1}y_2^{-1}(1\oplus y_2) \\ y_2(7) = y_1^3y_2^2(1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus z_1y_1y_2\oplus 2z_1y_1y_2^2\oplus z_1y_1y_2^3 \\ \qquad \oplus z_2y_1^2y_2^2\oplus z_2y_1^2y_2^3\oplus y_1^3y_2^3)^{-1} \end{cases}$$

$$\begin{cases} x_1(8) = x_1 \\ x_2(8) = x_1x_2^{-1} \frac{1+\hat{y}_2}{1\oplus y_2} \end{cases} \qquad \begin{cases} y_1(8) = y_1y_2(1\oplus y_2)^{-1} \\ y_2(8) = y_2^{-1} \end{cases}$$

$$\begin{cases} x_1(9) = x_1 \\ x_2(9) = x_2 \end{cases} \qquad \begin{cases} y_1(9) = y_1 \\ y_2(9) = y_2 \end{cases}$$

Table 1: Cluster variables and coefficients for the mutation sequence (3.7).

Proposition 3.8. *(cf. [N, Prop. 3.8]) The c-vectors satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$c_{ij}^{t'} = \begin{cases} -c_{ik}^t & \text{if } j = k; \\ c_{ij}^t + c_{ik}^t[d_k b_{kj}^t]_+ + [-c_{ik}^t]_+ d_k b_{kj}^t & \text{if } j \neq k. \end{cases} \tag{3.8}$$

Remark 3.9. It immediately follows that the c-vectors of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ do not depend on the choice of exchange polynomials $\mathbf{Z}$, only their degrees.

As in section 2 the X-functions become particularly nice.

Proposition 3.10. *(cf. [N, Prop. 3.3]) Each X-function X_i^t is contained in $\mathbb{Z}[\mathbf{x}^{\pm 1}, \mathbf{y}, \mathbf{z}]$.*

Using essentially the same $\mathbb{Z}^n$-grading these X-functions will once again be homogeneous.

Proposition 3.11. *(cf. [N, Prop. 3.15]) Under the $\mathbb{Z}^n$-grading*

$$\deg(x_i) = \mathbf{e}_i, \qquad \deg(y_j) = -\mathbf{b}_j, \qquad \text{and} \qquad \deg(z_{i,s}) = \mathbf{0},$$

each X-function is homogeneous and we write $\deg\left(X_j^t\right) = \mathbf{g}_j^t = \sum_{i=1}^n g_{ij}^t \mathbf{e}_i$. *Moreover, these g-vectors satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$g_{ij}^{t'} = \begin{cases} g_{ij}^t & \text{if } j \neq k; \\ -g_{ik}^t + \sum_{\ell=1}^n g_{i\ell}^t[-b_{\ell k}^t d_k]_+ - \sum_{\ell=1}^n b_{i\ell}^t[-c_{\ell k}^t d_k]_+ & \text{if } j = k. \end{cases}$$

Remark 3.12. It immediately follows that the g-vectors of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ also do not depend on the choice of exchange polynomials $\mathbf{Z}$, only their degrees.

Continuing to follow the developments of section 2 we may define F-polynomials $F_i^t(\mathbf{y}, \mathbf{z}) \in \mathbb{Z}[\mathbf{y}, \mathbf{z}]$ by specializing all cluster variables x_i to 1 in the X-functions, i.e. $F_i^t(\mathbf{y}, \mathbf{z}) = X_i^t(\mathbf{1}, \mathbf{y}, \mathbf{z})$.

Proposition 3.13. *(cf. [N, Prop. 3.12]) The F-polynomials satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$F_j^{t'} = \begin{cases} F_j^t & \text{if } j \neq k; \\ \left(F_k^t\right)^{-1} \left(\prod_{i=1}^n y_i^{[-c_{ik}^t]_+} \left(F_i^t\right)^{[-b_{ik}^t]_+} \right)^{d_k} Z_k \left(\prod_{i=1}^n y_i^{c_{ik}^t} \left(F_i^t\right)^{b_{ik}^t} \right) & \text{if } j = k. \end{cases} \tag{3.9}$$

The coefficients y_j^t can still be computed using the c-vectors and F-polynomials.

Theorem 3.14. *(cf. [N, Th. 3.23]) Fix an initial generalized seed* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *over a semifield* $\mathbb{P}$*. For any vertex* $t \in \mathbb{T}_n$ *each coefficient* y_j^t *of* $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *can be computed as*

$$y_j^t = \left(\prod_{i=1}^{n} y_i^{c_{ij}^t} \right) \prod_{i=1}^{n} F_i^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z})^{b_{ij}^t}. \tag{3.10}$$

Finally the separation of additions formula still holds for cluster variables of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$.

Theorem 3.15. *(cf. [N, Th. 3.24]) Fix an initial generalized seed* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *over a semifield* $\mathbb{P}$*. For any vertex* $t \in \mathbb{T}_n$ *each cluster variable* x_j^t *of* $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *can be computed as*

$$x_j^t = \left(\prod_{i=1}^{n} x_i^{g_{ij}^t} \right) \frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}}, \mathbf{z})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z})}, \tag{3.11}$$

where $\hat{y}_k = y_k \prod_{i=1}^{n} x_i^{b_{ik}}$.

Example 3.16. Following Theorems 3.14 and 3.15 we may immediately extract the C-matrix, G-matrix, and F-polynomials associated to each of the seeds $\Sigma(t)$ in Example 3.7. Writing $C(t)$, $G(t)$, and $F(t)$ for these quantities associated to the generalized seed $\Sigma(t)$ we obtain Table 2.

4 Companion Cluster Algebras

Fix an initial generalized seed $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ over a semifield $\mathbb{P}$. Write $D = (d_i \delta_{ij})$ where d_i is the degree of the exchange polynomial Z_i.

Denote by ${}^L\mathbf{x} := \mathbf{x}^{1/\mathbf{d}}$ the collection $({}^Lx_1, \cdots, {}^Lx_n) := (x_1^{1/d_1}, \ldots, x_n^{1/d_n})$ in the extension field $\mathbb{QP}(\mathbf{x}^{1/\mathbf{d}})$ of $\mathbb{QP}(\mathbf{x})$. For clarity we also write ${}^L\mathbf{y} = \mathbf{y}$, i.e. ${}^Ly_j = y_j$. Define the *left-companion cluster algebra* ${}^L\!\mathcal{A}$ of $\mathcal{A}$ to be $\mathcal{A}({}^L\mathbf{x}, {}^L\mathbf{y}, DB) \subset \mathbb{QP}(\mathbf{x}^{1/\mathbf{d}})$. Write $({}^L\mathbf{x}^t, {}^L\mathbf{y}^t, {}^LB^t)$ for the seed associated to vertex $t \in \mathbb{T}_n$ in the construction of ${}^L\!\mathcal{A}$ and denote by ${}^L\mathbf{c}_j^t$, ${}^L\mathbf{g}_j^t$, and ${}^LF_j^t$ the c-vectors, g-vectors, and F-polynomials of ${}^L\!\mathcal{A}$.

Write ${}^R\mathbf{x} = \mathbf{x}$, i.e. ${}^Rx_i = x_i$, and denote the collection $({}^Ry_1, \ldots, {}^Ry_n) = (y_1^{d_1}, \ldots, y_n^{d_n})$ by ${}^R\mathbf{y} := \mathbf{y}^{\mathbf{d}}$. Define the *right-companion cluster algebra* ${}^R\!\mathcal{A}$ of $\mathcal{A}$ to be $\mathcal{A}({}^R\mathbf{x}, {}^R\mathbf{y}, BD) \subset \mathbb{QP}(\mathbf{x})$. Write $({}^R\mathbf{x}^t, {}^R\mathbf{y}^t, {}^RB^t)$ for the seed associated to vertex $t \in \mathbb{T}_n$ in the construction of ${}^R\!\mathcal{A}$ and denote by ${}^R\mathbf{c}_j^t$, ${}^R\mathbf{g}_j^t$, and ${}^RF_j^t$ the c-vectors, g-vectors, and F-polynomials of ${}^R\!\mathcal{A}$.

We immediately obtain the following result as a consequence of Proposition 2.4 and Proposition 3.8 (cf. [N, Props. 3.9 and 3.10]).

$$C(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} F_1(1) = 1 \\ F_2(1) = 1 \end{cases}$$

$$C(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} F_1(2) = 1 + z_1y_1 + z_2y_1^2 + y_1^3 \\ F_2(2) = 1 \end{cases}$$

$$C(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad G(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} F_1(3) = 1 + z_1y_1 + z_2y_1^2 + y_1^3 \\ F_2(3) = 1 + y_2 + z_1y_1y_2 + z_2y_1^2y_2 + y_1^3y_2 \end{cases}$$

$$C(4) = \begin{pmatrix} 1 & -3 \\ 0 & -1 \end{pmatrix}, \quad G(4) = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}, \quad \begin{cases} F_1(4) = 1 + 3y_2 + 3y_2^2 + y_2^3 + 2z_1y_1y_2 + 4z_1y_1y_2^2 + 2z_1y_1y_2^3 \\ \qquad\qquad +z_2y_1^2y_2 + z_1^2y_1^2y_2^2 + 3z_2y_1^2y_2^2 + z_1^2y_1^2y_2^3 + 2z_2y_1^2y_2^3 \\ \qquad\qquad +z_1z_2y_1^3y_2^2 + 2z_1z_2y_1^3y_2^3 + 3y_1^3y_2^2 + 2y_1^3y_2^3 \\ \qquad\qquad +z_1y_1^4y_2^2 + 2z_1y_1^4y_2^3 + z_2^2y_1^4y_2^3 + 2z_2y_1^5y_2^3 + y_1^6y_2^3 \\ F_2(4) = 1 + y_2 + z_1y_1y_2 + z_2y_1^2y_2 + y_1^3y_2 \end{cases}$$

$$C(5) = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}, \quad G(5) = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}, \quad \begin{cases} F_1(5) = 1 + 3y_2 + 3y_2^2 + y_2^3 + 2z_1y_1y_2 + 4z_1y_1y_2^2 + 2z_1y_1y_2^3 \\ \qquad\qquad +z_2y_1^2y_2 + z_1^2y_1^2y_2^2 + 3z_2y_1^2y_2^2 + z_1^2y_1^2y_2^3 + 2z_2y_1^2y_2^3 \\ \qquad\qquad +z_1z_2y_1^3y_2^2 + 2z_1z_2y_1^3y_2^3 + 3y_1^3y_2^2 + 2y_1^3y_2^3 \\ \qquad\qquad +z_1y_1^4y_2^2 + 2z_1y_1^4y_2^3 + z_2^2y_1^4y_2^3 + 2z_2y_1^5y_2^3 + y_1^6y_2^3 \\ F_2(5) = 1 + 2y_2 + y_2^2 + z_1y_1y_2 + z_1y_1y_2^2 + z_2y_1^2y_2^2 + y_1^3y_2^2 \end{cases}$$

$$C(6) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}, \quad G(6) = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}, \quad \begin{cases} F_1(6) = 1 + 3y_2 + 3y_2^2 + y_2^3 + z_1y_1y_2 + 2z_1y_1y_2^2 + z_1y_1y_2^3 \\ \qquad\qquad +z_2y_1^2y_2^2 + z_2y_1^2y_2^3 + y_1^3y_2^3 \\ F_2(6) = 1 + 2y_2 + y_2^2 + z_1y_1y_2 + z_1y_1y_2^2 + z_2y_1^2y_2^2 + y_1^3y_2^2 \end{cases}$$

$$C(7) = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}, \quad G(7) = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}, \quad \begin{cases} F_1(7) = 1 + 3y_2 + 3y_2^2 + y_2^3 + z_1y_1y_2 + 2z_1y_1y_2^2 + z_1y_1y_2^3 \\ \qquad\qquad +z_2y_1^2y_2^2 + z_2y_1^2y_2^3 + y_1^3y_2^3 \\ F_2(7) = 1 + y_2 \end{cases}$$

$$C(8) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad G(8) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} F_1(8) = 1 \\ F_2(8) = 1 + y_2 \end{cases}$$

$$C(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} F_1(9) = 1 \\ F_2(9) = 1 \end{cases}$$

Table 2: C-matrices, G-matrices, and F-polynomials for the mutation sequence (3.7).

Corollary 4.1. *The c-vectors of the generalized cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y},B,\mathbf{Z})$ coincide with the c-vectors of its left-companion cluster algebra $\mathcal{A}(\mathbf{x}^{1/\mathbf{d}},\mathbf{y},DB)$ while the c-vectors of its right-companion cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y}^{\mathbf{d}},BD)$ can be obtained from those of $\mathcal{A}(\mathbf{x},\mathbf{y},B,\mathbf{Z})$ by the transformation ${}^{R}\mathbf{c}_j^t = d_i^{-1}c_{ij}^t d_j$.*

Similarly the following result is an immediate consequence of Proposition 2.7 and Proposition 3.11 (cf. [N, Props. 3.16 and 3.17]).

Corollary 4.2. *The g-vectors of the generalized cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y},B,\mathbf{Z})$ coincide with the g-vectors of its right-companion cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y}^{\mathbf{d}},BD)$ while the g-vectors of its left-companion cluster algebra $\mathcal{A}(\mathbf{x}^{1/\mathbf{d}},\mathbf{y},DB)$ can be obtained from those of $\mathcal{A}(\mathbf{x},\mathbf{y},B,\mathbf{Z})$ by the transformation ${}^{L}\mathbf{g}_j^t = d_i g_{ij}^t d_j^{-1}$.*

We see from Corollary 4.1 and Corollary 4.2 that the c- and g-vectors of the generalized cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y},B,\mathbf{Z})$ are intimately related to those of its left- and right-companion cluster algebras. The same is true for F-polynomials, however the precise relationship for left- and right-companions are very different.

We begin with the left-companion. For $1 \le i \le n$ and $0 \le s \le d_i$ we will write $\mathbf{z}^{\mathrm{bin}} = (z_{i,s}^{\mathrm{bin}})$ where $z_{i,s}^{\mathrm{bin}} = \binom{d_i}{s} \in \mathbb{Z}$.

Proposition 4.3. *Let $(\mathbf{x},\mathbf{y},B,\mathbf{Z})$ be a generalized seed over $\mathbb{P}$. For any $t \in \mathbb{T}_n$ and any $1 \le j \le n$ we have the following equalities in $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ and $\mathbb{Q}_{\mathrm{sf}}(\mathbf{x},\mathbf{y})$ respectively:*

$$F_j^t(\mathbf{y},\mathbf{z}^{\mathrm{bin}}) = {}^{L}F_j^t\big({}^{L}\mathbf{y}\big)^{d_j} \quad \textit{and} \quad F_j^t(\hat{\mathbf{y}},\mathbf{z}^{\mathrm{bin}}) = {}^{L}F_j^t\big({}^{L}\hat{\mathbf{y}}\big)^{d_j}, \tag{4.1}$$

where ${}^{L}y_i = y_i$ and ${}^{L}\hat{y}_i = \hat{y}_i$.

Proof. We will proceed by induction on the distance from t_0 to t in $\mathbb{T}_n$. To begin, note that by definition we have $\big({}^{L}x_j^{t_0}\big)^{d_j} = \big(x_j^{1/d_j}\big)^{d_j} = x_j = x_j^{t_0}$ so that $F_j^{t_0} = 1 = {}^{L}F_j^{t_0}$, in particular $F_j^{t_0} = \big({}^{L}F_j^{t_0}\big)^{d_j}$. Consider $t \overset{k}{-\!\!-} t'$ with t' further from t_0 than t and suppose $F_j^t = \big({}^{L}F_j^t\big)^{d_j}$ for all j. Then by Proposition 2.8 and Proposition 3.13 we see for $j \ne k$ that $F_j^{t'} = F_j^t = \big({}^{L}F_j^t\big)^{d_j} = \big({}^{L}F_j^{t'}\big)^{d_j}$ while taking $j = k$ we have

$$\begin{aligned}
{}^{L}F_k^{t'}({}^{L}\mathbf{y})^{d_k} &= \bigg(({}^{L}F_k^t)^{-1}\Big(\prod_{i=1}^n {}^{L}y_i^{[-{}^{L}c_{ik}^t]_+}({}^{L}F_i^t)^{[-d_i b_{ik}^t]_+}\Big)\Big(1+\prod_{i=1}^n {}^{L}y_i^{{}^{L}c_{ik}^t}({}^{L}F_i^t)^{d_i b_{ik}^t}\Big)\bigg)^{d_k}\\
&= \Big(({}^{L}F_k^t)^{d_k}\Big)^{-1}\Big(\prod_{i=1}^n {}^{L}y_i^{[-{}^{L}c_{ik}^t]_+}\big(({}^{L}F_i^t)^{d_i}\big)^{[-b_{ik}^t]_+}\Big)^{d_k}\sum_{s=0}^{d_k}\binom{d_k}{s}\Big(\prod_{i=1}^n {}^{L}y_i^{{}^{L}c_{ik}^t}\big(({}^{L}F_i^t)^{d_i}\big)^{b_{ik}^t}\Big)^s\\
&= (F_k^t)^{-1}\Big(\prod_{i=1}^n y_i^{[-c_{ik}^t]_+}(F_i^t)^{[-b_{ik}^t]_+}\Big)^{d_k}\sum_{s=0}^{d_k}\binom{d_k}{s}\Big(\prod_{i=1}^n y_i^{c_{ik}^t}(F_i^t)^{b_{ik}^t}\Big)^s\\
&= F_k^{t'}(\mathbf{y},\mathbf{z}^{\mathrm{bin}}),
\end{aligned}$$

where we used Corollary 4.1 in the third equality above. It follows by induction that $F_j^t(\mathbf{y},\mathbf{z}^{\mathrm{bin}}) = {}^{L}F_j^t\big({}^{L}\mathbf{y}\big)^{d_j}$ for all $t \in \mathbb{T}_n$ and $1 \le j \le n$. Note that $\hat{y}_j = y_j \prod_{i=1}^{n}$

$x_i^{b_{ij}} = {}^L y_j \prod_{i=1}^n {}^L x_i^{d_i b_{ij}} = {}^L \hat{y}_j$ so that substituting the variables $\hat{y}_j$ into this identity gives $F_j^t(\hat{\mathbf{y}}, \mathbf{z}^{\mathrm{bin}}) = {}^L F_j^t({}^L \hat{\mathbf{y}})^{d_j}$ for all $t \in \mathbb{T}_n$ and $1 \le j \le n$. □

Write $x_i^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}} \in \mathcal{F}$ and $y_j^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}} \in \mathbb{P}$ for the variables obtained by applying equations (3.15) and (3.14) respectively using the specialized F-polynomials $F_j^t(\mathbf{y}, \mathbf{z}^{\mathrm{bin}})$ in place of the generic F-polynomials $F_j^t(\mathbf{y}, \mathbf{z})$.

Theorem 4.4. *We have* $x_i^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}} = ({}^L x_i^t)^{d_i}$ *and* $y_j^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}} = {}^L y_i^t$.

Proof. For coefficients we apply Theorem 2.9 and Theorem 3.14 along with Corollary 4.1 to get

$$ {}^L y_j^t = \left(\prod_{i=1}^n {}^L y_i^{{}^L c_{ij}^t} \right) \prod_{i=1}^n {}^L F_i^t\big|_{\mathbb{P}}({}^L \mathbf{y})^{d_i b_{ij}^t} = \left(\prod_{i=1}^n y_i^{c_{ij}^t} \right) \prod_{i=1}^n F_i^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z}^{\mathrm{bin}})^{b_{ij}^t} = y_j^t. $$

To finish, we may apply Theorem 2.10 and Theorem 3.15 along with Corollary 4.2 to get

$$ \begin{aligned} ({}^L x_j^t)^{d_j} &= \left(\left(\prod_{i=1}^n {}^L x_i^{{}^L g_{ij}^t} \right) \frac{{}^L F_j^t\big|_{\mathcal{F}}({}^L \hat{\mathbf{y}})}{{}^L F_j^t\big|_{\mathbb{P}}({}^L \mathbf{y})} \right)^{d_j} = \left(\prod_{i=1}^n {}^L x_i^{{}^L g_{ij}^t d_j} \right) \frac{{}^L F_j^t\big|_{\mathcal{F}}({}^L \hat{\mathbf{y}})^{d_j}}{{}^L F_j^t\big|_{\mathbb{P}}({}^L \mathbf{y})^{d_j}} \\ &= \left(\prod_{i=1}^n \left({}^L x_i^{1/d_i}\right)^{g_{ij}^t} \right) \frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}}, \mathbf{z}^{\mathrm{bin}})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z}^{\mathrm{bin}})} = \left(\prod_{i=1}^n x_i^{g_{ij}^t} \right) \frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}}, \mathbf{z}^{\mathrm{bin}})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z}^{\mathrm{bin}})} = x_j^t. \end{aligned} $$

□

Example 4.5. As an illustration of Corollaries 4.1 and 4.2 as well as Theorem 4.4 we now present the C-matrices, G-matrices, and F-polynomials for the left companion cluster algebra ${}^L\mathcal{A}$ in Table 3 from which we invite the reader to directly verify these results.

To state a relationship between a generalized cluster algebra and its right-companion we need the following analogue of Proposition 4.3.

Proposition 4.6. *Let* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *be a generalized seed over* $\mathbb{P}$. *For any* $t \in \mathbb{T}_n$ *and any* $1 \le j \le n$ *we have the following equalities in* $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ *and* $\mathbb{Q}_{\mathrm{sf}}(\mathbf{x}, \mathbf{y})$ *respectively:*

$$ F_j^t(\mathbf{y}, \mathbf{0}) = {}^R F_j^t({}^R \mathbf{y}) \quad \textit{and} \quad F_j^t(\hat{\mathbf{y}}, \mathbf{0}) = {}^R F_j^t({}^R \hat{\mathbf{y}}), \tag{4.2} $$

where ${}^R y_i = y_i^{d_i}$ *and* ${}^R \hat{y}_i = \hat{y}_i^{d_i}$.

$$
\begin{aligned}
&{}^L C(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^L G(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^L F_1(1) = 1 \\ {}^L F_2(1) = 1 \end{cases} \\
&{}^L C(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^L G(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^L F_1(2) = 1 + {}^L y_1 \\ {}^L F_2(2) = 1 \end{cases} \\
&{}^L C(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad {}^L G(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} {}^L F_1(3) = 1 + {}^L y_1 \\ {}^L F_2(3) = 1 + {}^L y_2 + 3 {}^L y_1 {}^L y_2 + 3 {}^L y_1^2 {}^L y_2 + {}^L y_1^3 {}^L y_2 \end{cases} \\
&{}^L C(4) = \begin{pmatrix} 1 & -3 \\ 0 & -1 \end{pmatrix}, \quad {}^L G(4) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \begin{cases} {}^L F_1(4) = 1 + {}^L y_2 + 2 {}^L y_1 {}^L y_2 + {}^L y_1^2 {}^L y_2 \\ {}^L F_2(4) = 1 + {}^L y_2 + 3 {}^L y_1 {}^L y_2 + 3 {}^L y_1^2 {}^L y_2 + {}^L y_1^3 {}^L y_2 \end{cases} \\
&{}^L C(5) = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}, \quad {}^L G(5) = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}, \quad \begin{cases} {}^L F_1(5) = 1 + {}^L y_2 + 2 {}^L y_1 {}^L y_2 + {}^L y_1^2 {}^L y_2 \\ {}^L F_2(5) = 1 + 2 {}^L y_2 + {}^L y_2^2 + 3 {}^L y_1 {}^L y_2 \\ \qquad\qquad + 3 {}^L y_1 {}^L y_2^2 + 3 {}^L y_1^2 {}^L y_2^2 + {}^L y_1^3 {}^L y_2^2 \end{cases} \\
&{}^L C(6) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}, \quad {}^L G(6) = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}, \quad \begin{cases} {}^L F_1(6) = 1 + {}^L y_2 + {}^L y_1 {}^L y_2 \\ {}^L F_2(6) = 1 + 2 {}^L y_2 + {}^L y_2^2 + 3 {}^L y_1 {}^L y_2 \\ \qquad\qquad + 3 {}^L y_1 {}^L y_2^2 + 3 {}^L y_1^2 {}^L y_2^2 + {}^L y_1^3 {}^L y_2^2 \end{cases} \\
&{}^L C(7) = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}, \quad {}^L G(7) = \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}, \quad \begin{cases} {}^L F_1(7) = 1 + {}^L y_2 + {}^L y_1 {}^L y_2 \\ {}^L F_2(7) = 1 + {}^L y_2 \end{cases} \\
&{}^L C(8) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad {}^L G(8) = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} {}^L F_1(8) = 1 \\ {}^L F_2(8) = 1 + {}^L y_2 \end{cases} \\
&{}^L C(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^L G(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^L F_1(9) = 1 \\ {}^L F_2(9) = 1 \end{cases}
\end{aligned}
$$

Table 3: C-matrices, G-matrices, and F-polynomials for the same mutation sequence (3.7) applied to the seeds of ${}^L\mathcal{A}$.

Proof. We will proceed by induction on the distance from t_0 to t in $\mathbb{T}_n$. To begin, note that by definition we have $F_j^{t_0} = 1 = {}^R\!F_j^{t_0}$. Consider $t \overset{k}{-\!\!-} t'$ with t' further from t_0 than t and suppose $F_j^t = {}^R\!F_j^t$ for all j. Then by Proposition 2.8 and Proposition 3.13 we see for $j \neq k$ that $F_j^{t'} = F_j^t = {}^R\!F_j^t = {}^R\!F_j^{t'}$ while taking $j = k$ we have

$$\begin{aligned}
{}^R\!F_k^{t'}\big({}^R\mathbf{y}\big) &= \big({}^R\!F_k^t\big)^{-1}\left(\prod_{i=1}^n {}^R y_i^{[-{}^Rc_{ik}^t]_+}\big({}^R\!F_i^t\big)^{[-b_{ik}^t d_k]_+}\right)\left(1+\prod_{i=1}^n {}^R y_i^{{}^Rc_{ik}^t}\big({}^R\!F_i^t\big)^{b_{ik}^t d_k}\right)\\
&= \big(F_k^t\big)^{-1}\left(\prod_{i=1}^n y_i^{[-c_{ik}^t d_k]_+}\big(F_i^t\big)^{[-b_{ik}^t d_k]_+}\right)\left(1+\prod_{i=1}^n y_i^{c_{ik}^t d_k}\big(F_i^t\big)^{b_{ik}^t d_k}\right)\\
&= \big(F_k^t\big)^{-1}\left(\prod_{i=1}^n y_i^{[-c_{ik}^t]_+}\big(F_i^t\big)^{[-b_{ik}^t]_+}\right)^{d_k}\left(1+\left(\prod_{i=1}^n y_i^{c_{ik}^t}\big(F_i^t\big)^{b_{ik}^t}\right)^{d_k}\right)\\
&= F_k^{t'}(\mathbf{y},\mathbf{0}).
\end{aligned}$$

It follows by induction that $F_j^t(\mathbf{y},\mathbf{0}) = {}^R\!F_j^t\big({}^R\mathbf{y}\big)$ for all $t \in \mathbb{T}_n$ and $1 \le j \le n$. Finally notice that ${}^R\hat{y}_j = {}^R y_j \prod_{i=1}^n {}^R x_i^{b_{ij}d_j} = y_j^{d_j}\prod_{i=1}^n x_i^{b_{ij}d_j} = \hat{y}_j^{d_j}$ so that substituting the variables $\hat{y}_j$ into this identity gives $F_j^t(\hat{\mathbf{y}},\mathbf{0}) = {}^R\!F_j^t\big({}^R\hat{\mathbf{y}}\big)$ for all $t \in \mathbb{T}_n$ and $1 \le j \le n$. □

Write $x_i^t\big|_{\mathbf{z}=\mathbf{0}} \in \mathcal{F}$ and $y_j^t\big|_{\mathbf{z}=\mathbf{0}} \in \mathbb{P}$ for the variables obtained by applying equations (3.15) and (3.14) respectively using the specialized F-polynomials $F_j^t(\mathbf{y},\mathbf{0})$ in place of the generic F-polynomials $F_j^t(\mathbf{y},\mathbf{z})$.

Theorem 4.7. *We have* $x_i^t\big|_{\mathbf{z}=\mathbf{0}} = {}^R x_i^t$ *and* $\big(y_j^t\big|_{\mathbf{z}=\mathbf{0}}\big)^{d_j} = {}^R y_j^t$.

Proof. To see the claim for coefficients we apply Theorem 2.9 and Theorem 3.14 along with Corollary 4.1 and Proposition 4.6 to get

$${}^R y_j^t = \left(\prod_{i=1}^n {}^R y_i^{{}^Rc_{ij}^t}\right)\prod_{i=1}^n {}^R\!F_i^t\big|_{\mathbb{P}}\big({}^R\mathbf{y}\big)^{b_{ij}^t d_j} = \left(\prod_{i=1}^n y_i^{c_{ij}^t d_j}\right)\prod_{i=1}^n F_i^t\big|_{\mathbb{P}}(\mathbf{y},\mathbf{0})^{b_{ij}^t d_j} = \big(y_j^t\big)^{d_j}.$$

Finally to see the claim for cluster variables we apply Theorem 2.10 and Theorem 3.15 along with Corollary 4.2 and Proposition 4.6 to get

$${}^R x_j^t = \left(\prod_{i=1}^n {}^R x_i^{{}^Rg_{ij}^t}\right)\frac{{}^R\!F_j^t\big|_{\mathcal{F}}\big({}^R\hat{\mathbf{y}}\big)}{{}^R\!F_j^t\big|_{\mathbb{P}}\big({}^R\mathbf{y}\big)} = \left(\prod_{i=1}^n x_i^{g_{ij}^t}\right)\frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}},\mathbf{0})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y},\mathbf{0})} = x_j^t.$$

□

Example 4.8. As an illustration of Corollaries 4.1 and 4.2 as well as Theorem 4.7 we now present the C-matrices, G-matrices, and F-polynomials for the right companion cluster algebra ${}^R\!\mathcal{A}$ in Table 4 from which we invite the reader to directly verify these results.

$$ {}^{R}C(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^{R}G(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(1) = 1 \\ {}^{R}F_2(1) = 1 \end{cases} $$

$$ {}^{R}C(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^{R}G(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(2) = 1 + {}^{R}y_1 \\ {}^{R}F_2(2) = 1 \end{cases} $$

$$ {}^{R}C(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad {}^{R}G(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(3) = 1 + {}^{R}y_1 \\ {}^{R}F_2(3) = 1 + {}^{R}y_2 + {}^{R}y_1{}^{R}y_2 \end{cases} $$

$$ {}^{R}C(4) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad {}^{R}G(4) = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(4) = 1 + 3{}^{R}y_2 + 3{}^{R}y_2^2 + {}^{R}y_2^3 \\ \qquad\qquad +3{}^{R}y_1{}^{R}y_2^2 + 2{}^{R}y_1{}^{R}y_2^3 + {}^{R}y_1^2{}^{R}y_2^3 \\ {}^{R}F_2(4) = 1 + {}^{R}y_2 + {}^{R}y_1{}^{R}y_2 \end{cases} $$

$$ {}^{R}C(5) = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}, \quad {}^{R}G(5) = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(5) = 1 + 3{}^{R}y_2 + 3{}^{R}y_2^2 + {}^{R}y_2^3 \\ \qquad\qquad +3{}^{R}y_1{}^{R}y_2^2 + 2{}^{R}y_1{}^{R}y_2^3 + {}^{R}y_1^2{}^{R}y_2^3 \\ {}^{R}F_2(5) = 1 + 2{}^{R}y_2 + {}^{R}y_2^2 + {}^{R}y_1{}^{R}y_2^2 \end{cases} $$

$$ {}^{R}C(6) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \quad {}^{R}G(6) = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(6) = 1 + 3{}^{R}y_2 + 3{}^{R}y_2^2 + {}^{R}y_2^3 + {}^{R}y_1{}^{R}y_2^3 \\ {}^{R}F_2(6) = 1 + 2{}^{R}y_2 + {}^{R}y_2^2 + {}^{R}y_1{}^{R}y_2^2 \end{cases} $$

$$ {}^{R}C(7) = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}, \quad {}^{R}G(7) = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(7) = 1 + 3{}^{R}y_2 + 3{}^{R}y_2^2 + {}^{R}y_2^3 + {}^{R}y_1{}^{R}y_2^3 \\ {}^{R}F_2(7) = 1 + {}^{R}y_2 \end{cases} $$

$$ {}^{R}C(8) = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \quad {}^{R}G(8) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(8) = 1 \\ {}^{R}F_2(8) = 1 + {}^{R}y_2 \end{cases} $$

$$ {}^{R}C(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^{R}G(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^{R}F_1(9) = 1 \\ {}^{R}F_2(9) = 1 \end{cases} $$

Table 4: C-matrices, G-matrices, and F-polynomials for the same mutation sequence (3.7) applied to the seeds of ${}^{R}\mathcal{A}$.

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