Travaux mathématiques, Volume 24 (2016), 167–181,

Yang–Mills–Higgs connections on Calabi–Yau manifolds, II

by Indranil Biswas, Ugo Bruzzo,
Beatriz Graña Otero, and Alessio Lo Giudice [1]

Abstract

In this paper we study Higgs and co–Higgs G–bundles on compact Kähler manifolds X. Our main results are:

1. If X is Calabi–Yau (i.e., it has vanishing first Chern class), and (E, θ) is a semistable Higgs or co–Higgs G–bundle on X, then the principal G–bundle E is semistable. In particular, there is a deformation retract of $\mathcal{M}_H(G)$ onto $\mathcal{M}(G)$, where $\mathcal{M}(G)$ is the moduli space of semistable principal G–bundles with vanishing rational Chern classes on X, and analogously, $\mathcal{M}_H(G)$ is the moduli space of semistable principal Higgs G–bundles with vanishing rational Chern classes.

2. Calabi–Yau manifolds are characterized as those compact Kähler manifolds whose tangent bundle is semistable for every Kähler class, and have the following property: if (E, θ) is a semistable Higgs or co–Higgs vector bundle, then E is semistable.

1 Introduction

In our previous paper [BBGL] we showed that the existence of semistable Higgs bundles with a nontrivial Higgs field on a compact Kähler manifold X constrains the geometry of X. In particular, it was shown that if X is Kähler-Einstein with $c_1(TX) \geq 0$, then it is necessarily Calabi-Yau, i.e., $c_1(TX) = 0$. In this paper we extend the analysis of the interplay between the existence of semistable Higgs bundles and the geometry of the underlying manifold (actually, we shall

Keywords: Higgs G–bundle, co–Higgs bundle, Yang–Mills–Higgs connection, representation space, deformation retraction.

AMS Classification: 14F05, 14J32, 32L05, 53C07, 58E15

[1]I.B. thanks Université Pierre et Marie Curie for hospitality. He is supported by a J. C. Bose Fellowship. U.B.'s research is partly supported by INdAM-GNSAGA. B.G.O. is supported by Project ID PRY 6777 of the Pontificia Universidad Javeriana, Bogotá. U.B. is a member of VBAC.

also consider co-Higgs bundles, and allow the structure group of the bundle to be any reductive linear algebraic group). Thus, if X is Calabi–Yau and (E, θ) is a semistable Higgs or co–Higgs G–bundle on X, it is proved that the underlying principal G–bundle E is semistable (Lemma 5.1). This has a consequence on the topology of the moduli spaces of principal (Higgs) G-bundles having vanishing rational Chern classes. We can indeed prove that there is a deformation retract of $\mathcal{M}_H(G)$ onto $\mathcal{M}(G)$, where $\mathcal{M}(G)$ is the moduli space of semistable principal G–bundles with vanishing rational Chern classes, and analogously, $\mathcal{M}_H(G)$ is the moduli space of semistable principal Higgs G–bundles with vanishing rational Chern classes (cf. [BF, FL] for similar deformation retract results).

As a further application, we can prove a characterization of Calabi–Yau manifolds in terms of Higgs and co-Higgs bundles; the characterization in question says that if X is a compact Kähler manifold with semistable tangent bundle with respect to every Kähler class, having the following property: for any semistable Higgs or co–Higgs vector bundle (E, θ) on X, the vector bundle E is semistable, then X is Calabi-Yau (Theorem 5.2).

In Section 4 We give a result about the behavior of semistable Higgs bundles under pullback by finite morphisms of Kähler manifolds. Let (X, ω) be a Ricci–flat compact Kähler manifold, M a compact connected Kähler manifold, and

$$f : M \longrightarrow X$$

a surjective holomorphic map such that each fiber of f is a finite subset of M. Let (E_G, θ) be a Higgs G–bundle on X such that the pulled back Higgs G–bundle $(f^*E_G, f^*\theta)$ on M is semistable (respectively, stable). Then the principal G–bundle f^*E_G is semistable (respectively, polystable).

2 Preliminaries

Let X be a compact connected Kähler manifold equipped with a Kähler form ω. Using ω, the degree of torsion-free coherent analytic sheaves on X is defined as follows:

$$\text{degree}(F) := \int_X c_1(F) \bigwedge \omega^{d-1} \in \mathbb{R}\,,$$

where $d = \dim_{\mathbb{C}} X$. The holomorphic cotangent bundle of X will be denoted by Ω_X.

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. The connected component of the center of G containing the identity element will be denoted by $Z_0(G)$. The Lie algebra of G will be denoted by $\mathfrak{g}$. A Zariski closed connected subgroup $P \subseteq G$ is called a parabolic subgroup of G if G/P is a projective variety. The unipotent radical of a parabolic subgroup P will be denoted by $R_u(P)$. A Levi subgroup of a parabolic subgroup P is a Zariski closed subgroup

$L(P) \subset P$ such that the composition

$$L(P) \hookrightarrow P \longrightarrow P/R_u(P)$$

is an isomorphism. Levi subgroups exist, and any two Levi subgroups of P differ by an inner automorphism of P [Bo, § 11.22, p. 158], [Hu2, § 30.2, p. 184]. The quotient map $G \longrightarrow G/P$ defines a principal P–bundle on G/P. The holomorphic line bundle on G/P associated to this principal P–bundle for a character χ of P will be denoted by $G(\chi)$. A character χ of a parabolic subgroup P is called *strictly anti–dominant* if $\chi|_{Z_0(G)}$ is trivial, and the associated holomorphic line bundle on $G(\chi) \longrightarrow G/P$ is ample.

For a principal G–bundle E_G on X, the vector bundle

$$\mathrm{ad}(E_G) := E_G \times^G \mathfrak{g} \longrightarrow X$$

associated to E_G for the adjoint action of G on its Lie algebra $\mathfrak{g}$ is called the *adjoint bundle* for E_G. So the fibers of $\mathrm{ad}(E_G)$ are Lie algebras identified with $\mathfrak{g}$ up to inner automorphisms. Using the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$ and the exterior multiplication of differential forms we have a symmetric bilinear pairing

$$(\mathrm{ad}(E_G) \otimes \Omega_X) \times (\mathrm{ad}(E_G) \otimes \Omega_X) \longrightarrow \mathrm{ad}(E_G) \otimes \Omega_X^2$$

which will be denoted by $\bigwedge$.

A *Higgs field* on a holomorphic principal G–bundle E_G on X is a holomorphic section θ of $\mathrm{ad}(E_G) \otimes \Omega_X$ such that

$$\theta \bigwedge \theta = 0 \,. \tag{2.1}$$

A *Higgs G–bundle* on X is a pair of the form (E_G, θ), where E_G is holomorphic principal G–bundle on X and θ is a Higgs field on E_G. A Higgs G–bundle (E_G, θ) is called *stable* (respectively, *semistable*) if for every quadruple of the form (U, P, χ, E_P), where

- $U \subset X$ is a dense open subset such that the complement $X \setminus U$ is a complex analytic subset of X of complex codimension at least two,
- $P \subset G$ is a proper parabolic subgroup,
- χ is a strictly anti–dominant character of P, and
- $E_P \subset E_G|_U$ is a holomorphic reduction of structure group to P over U such that $\theta|_U$ is a section of $\mathrm{ad}(E_P) \otimes \Omega_U$,

the following holds:

$$\text{degree}(E_P \times^{\chi} \mathbb{C}) \,>\, 0$$

(respectively, degree$(E_P \times^{\chi} \mathbb{C}) \geq 0$); note that since $X \setminus U$ is a complex analytic subset of X of complex codimension at least two, the line bundle $E_P \times^{\chi} \mathbb{C}$ on U extends uniquely to a holomorphic line bundle on X.

A semistable Higgs G–bundle (E_G, θ) is called *polystable* if there is a Levi subgroup $L(Q)$ of a parabolic subgroup $Q \subset G$ and a Higgs $L(Q)$–bundle (E', θ') on X such that

- the Higgs G–bundle obtained by extending the structure group of (E', θ') using the inclusion $L(Q) \hookrightarrow G$ is isomorphic to (E_G, θ), and
- the Higgs $L(Q)$–bundle (E', θ') is stable.

Fix a maximal compact subgroup $K \subset G$. Given a holomorphic principal G–bundle E_G and a C^∞ reduction of structure group $E_K \subset E_G$ to the subgroup K, there is a unique connection on the principal K–bundle E_K that is compatible with the holomorphic structure of E_G [At, pp. 191–192, Proposition 5]; it is known as the *Chern connection.* A C^∞ reduction of structure group of E_G to K is called a *Hermitian structure* on E_G.

Let Λ_ω denote the adjoint of multiplication of differential forms on X by ω.

Given a Higgs G–bundle (E_G, θ) on X, a Hermitian structure $E_K \subset E_G$ is said to satisfy the Yang–Mills–Higgs equation if

$$\Lambda_\omega(\mathcal{K} + \theta \bigwedge \theta^*) \,=\, \mathfrak{z}\,, \tag{2.2}$$

where $\mathcal{K}$ is the curvature of the Chern connection associated to E_K and $\mathfrak{z}$ is some element of the Lie algebra of $Z_0(G)$; the adjoint θ^* is constructed using the Hermitian structure E_K. A Higgs G–bundle admits a Hermitian structure satisfying the Yang–Mills–Higgs equation if and only if it is polystable [Si], [BiSc, p. 554, Theorem 4.6].

Given a polystable Higgs G–bundle (E, θ), any two Hermitian structures on E_G satisfying the Yang–Mills–Higgs equation differ by a holomorphic automorphism of E_G that preserves θ; however, the associated Chern connection is unique [BiSc, p. 554, Theorem 4.6].

3 Higgs G–bundles on Calabi–Yau manifolds

Henceforth, till the end of Section 4, we assume that $c_1(TX) \in H^2(X, \mathbb{Q})$ is zero. From this assumption it follows that every Kähler class on X contains a Ricci–flat Kähler metric [Ya, p. 364, Theorem 2]. We will assume that the Kähler form ω on X is Ricci–flat.

3.1 Higgs G–bundles on Calabi-Yau manifolds

Let $(E_G,\, \theta)$ be a polystable Higgs G–bundle on X. For any holomorphic tangent vector $v \in T_xX$, note that $\theta(x)$ is an element of the fiber $\mathrm{ad}(E_G)_x$. For any point $x \in X$, consider the complex subspace

$$\widehat{\Theta}_x := \{\theta(x)(v) \mid v \in T_xX\} \subset \mathrm{ad}(E_G)_x\,. \tag{3.1}$$

Form (2.1) it follows immediately that $\widehat{\Theta}_x$ is an abelian subalgebra of the Lie algebra $\mathrm{ad}(E_G)_x$.

Let ∇ be the connection on $\mathrm{ad}(E_G)$ induced by the unique connection on E_G given by the solutions of the Yang–Mills–Higgs equation.

Lemma 3.1.

1. *The abelian subalgebra* $\widehat{\Theta}_x \subset \mathrm{ad}(E_G)_x$ *is semisimple.*
2. $\{\widehat{\Theta}_x\}_{x\in X} \subset \mathrm{ad}(E_G)$ *is preserved by the connection* ∇ *on* $\mathrm{ad}(E_G)$. *In particular,*
$$\{\widehat{\Theta}_x\}_{x\in X} \subset \mathrm{ad}(E_G)$$
is a holomorphic subbundle.

Proof. First take $G = \mathrm{GL}(n,\mathbb{C})$, so that $(E_G,\, \theta)$ defines a Higgs vector bundle $(F,\, \varphi)$ of rank n. Let
$$\widehat{\Theta}'_x \subset \mathrm{End}(F_x)$$
be the subalgebra constructed as in (3.1) for the Higgs vector bundle $(F,\, \varphi)$. From [BBGL, Proposition 2.5] it follows immediately that there is a basis of the vector space F_x such that
$$\varphi(x)(v) \in \mathrm{End}(F_x)$$
is diagonal with respect to it for all $v \in T_x$. This implies that the subalgebra $\widehat{\Theta}'_x$ is semisimple (this uses the Jordan–Chevalley decomposition, see e.g. [Hu1, Ch. 2]).

Consider the $\mathcal{O}_X$–linear homomorphism

$$\eta : TX \longrightarrow \mathrm{End}(F)$$

that sends any $w \in T_yX$ to $\varphi(y)(w) \in \mathrm{End}(F_y)$, where φ as before is the Higgs field on the holomorphic vector bundle F. Proposition 2.2 of [BBGL] says that φ is flat with respect to the connection on $\mathrm{End}(F) \otimes \Omega_X$ induced by the connection ∇ on $\mathrm{End}(F) = \mathrm{ad}(E_G)$ and the Levi–Civita connection on Ω_X for ω. Therefore, the above homomorphism η intertwines the Levi–Civita connection on TX and the connection on $\mathrm{End}(F)$. Consequently, the image $\eta(TX) \subset \mathrm{End}(F)$ is preserved by the connection on $\mathrm{End}(F)$. On the other hand, $\eta(TX)$ coincides with $\{\widehat{\Theta}'_x\}_{x\in X} \subset \mathrm{End}(F)$.

Therefore, the lemma is proved when $G = \mathrm{GL}(n, \mathbb{C})$.

For a general G, take any homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(N, \mathbb{C})$$

such that $\rho(Z_0(G))$ lies inside the center of $\mathrm{GL}(N, \mathbb{C})$. Let (E_ρ, θ_ρ) be the Higgs vector bundle of rank N given by (E_G, θ) using ρ. For any Hermitian structure on E_G solving the Yang–Mills–Higgs equation for (E_G, θ), the induced Hermitian structure on E_ρ solves the Yang–Mills–Higgs equation for (E_ρ, θ_ρ). We have shown above that the lemma holds for (E_ρ, θ_ρ).

Since the lemma holds for (E_ρ, θ_ρ) for every ρ of the above type, we conclude that the lemma holds for (E_G, θ). □

As before, (E_G, θ) is a polystable Higgs G–bundle on X. Fix a Hermitian structure

$$E_K \subset E_G \tag{3.2}$$

that satisfies the Yang–Mills–Higgs equation for (E_G, θ).

Take another Higgs field β on E_G. Let

$$\widetilde{\beta} : TX \longrightarrow \mathrm{ad}(E_G)$$

be the $\mathcal{O}_X$–linear homomorphism that sends any tangent vector $w \in T_yX$ to

$$\beta(y)(w) \in \mathrm{ad}(E_G)_y\,.$$

Theorem 3.2. *Assume that the image $\widetilde{\beta}(TX)$ is contained in the subbundle*

$$\{\widehat{\Theta}_x\}_{x\in X} \subset \mathrm{ad}(E_G)$$

in Lemma 3.1. Then E_K in (3.2) *also satisfies the Yang–Mills–Higgs equation for (E_G, β). In particular, (E_G, β) is polystable.*

Proof. From Theorem 4.2 of [BBGL] we know that E_K in (3.2) satisfies the Yang–Mills–Higgs equation for $(E_G, 0)$. Therefore, it suffices to show that $\beta \bigwedge \beta^* = 0$ (see (2.2)).

Let

$$\gamma : TX \longrightarrow \mathrm{ad}(E_G)$$

be the $C^\infty(X)$–linear homomorphism that sends any $w \in T_yX$ to $\theta^*(y)(w) \in \mathrm{ad}(E_G)_y$. Clearly, we have

$$\gamma(TX)^* = \{\widehat{\Theta}_x\}_{x\in X}\,; \tag{3.3}$$

as before, the superscript "$*$" denotes adjoint with respect to the reduction E_K. Since the subbundle $\{\widehat{\Theta}_x\}_{x\in X}$ is preserved by the connection on $\mathrm{ad}(E_G)$, from (3.3) it follows that

$$\{\widehat{\Theta}_x\}_{x\in X} + \gamma(TX) \subset \mathrm{ad}(E_G) \tag{3.4}$$

is a subbundle preserved by the connection; it should be clarified that the above need not be a direct sum.

We know that $\theta \bigwedge \theta^* = 0$ [BBGL, Lemma 4.1]. This and (2.1) together imply that the subbundle in (3.4) is an abelian subalgebra bundle. We have

$$\widetilde{\beta}(TX) \subset \{\widehat{\Theta}_x\}_{x\in X}\,,$$

and hence β^* is a section of $\gamma(TX)\otimes\Omega_X \subset \mathrm{ad}(E_G)\otimes\Omega_X$. Since the subbundle in (3.4) is an abelian subalgebra bundle, we now conclude that $\beta \bigwedge \beta^* = 0$. □

The proof of Theorem 3.2 gives the following:

Corollary 3.3. *Let ϕ be a Higgs field on E_G such that $\phi \bigwedge \phi^* = 0$. Then E_K in* (3.2) *also satisfies the Yang–Mills–Higgs equation for* $(E_G,\, \phi)$. *In particular,* $(E_G,\, \phi)$ *is polystable.*

Proof. As noted in the proof of Theorem 3.2, the reduction E_K satisfies the Yang–Mills–Higgs equation for $(E_G,\, 0)$. Since $\phi \bigwedge \phi^* = 0$, it follows that E_K in (3.2) satisfies the Yang–Mills–Higgs equation for $(E_G,\, \phi)$. □

Remark 3.4. The condition in Theorem 3.2 that $\widetilde{\beta}(TX) \subset \{\widehat{\Theta}_x\}_{x\in X}$ does not depend on the Hermitian structure E_K; it depends only on the Higgs G–bundle $(E_G,\, \theta)$. In contrast, the condition $\phi \bigwedge \phi^* = 0$ in Corollary 3.3 depends also on E_K.

3.2 A deformation retraction

Let $\mathcal{M}_H(G)$ denote the moduli space of semistable Higgs G–bundles $(E_G,\, \theta)$ on X such that all rational characteristic classes of E_G of positive degree vanish. It is known (it is a straightforward consequence of Theorem 2 in [Si]) that if the following three conditions hold:

1. $(E_G,\, \theta)$ is semistable,
2. for all characters χ of G, the line bundle on X associated to E_G for χ is of degree zero, and
3. the second Chern class $c_2(\mathrm{ad}(E_G)) \in H^4(X,\, \mathbb{Q})$ vanishes,

then all characteristic classes of E_G of positive degree vanish. Let $\mathcal{M}(G)$ denote the moduli space of semistable principal G–bundles E_G on X such that all rational characteristic classes of E_G of positive degree vanish.

We have an inclusion

$$\xi : \mathcal{M}(G) \longrightarrow \mathcal{M}_H(G), \quad E_G \longmapsto (E_G, 0). \tag{3.5}$$

Proposition 3.5. *There is a natural holomorphic deformation retraction of $\mathcal{M}_H(G)$ to the image of ξ in* (3.5)*.*

Proof. Points of $\mathcal{M}_H(G)$ parametrize the polystable Higgs G–bundles (E_G, θ) on X such that all rational characteristic classes of E_G of positive degree vanish. Given such a Higgs G–bundle (E_G, θ), from Theorem 3.2 we know that $(E_G, t\cdot\theta)$ is polystable for all $t \in \mathbb{C}$. Therefore, we have a holomorphic map

$$F : \mathbb{C} \times \mathcal{M}_H(G) \longrightarrow \mathcal{M}_H(G), \quad (t, (E_G, \theta)) \longmapsto (E_G, t\cdot\theta).$$

The restriction of F to $\{1\} \times \mathcal{M}_H(G)$ is the identity map of $\mathcal{M}_H(G)$, while the image of the restriction of F to $\{0\} \times \mathcal{M}_H(G)$ is the image of ξ. Moreover, the restriction of F to $\{0\} \times \xi(\mathcal{M}(G))$ is the identity map. □

Fix a point $x_0 \in X$. Since G is an affine variety and $\pi_1(X, x_0)$ is finitely presented, the geometric invariant theoretic quotient

$$\mathcal{M}_R(G) := \mathrm{Hom}(\pi_1(X, x_0), G)/\!\!/G$$

for the adjoint action of G on $\mathrm{Hom}(\pi_1(X, x_0), G)$ is an affine variety. The points of $\mathcal{M}_R(G)$ parameterize the equivalence classes of homomorphisms from $\pi_1(X, x_0)$ to G such that the Zariski closure of the image is a reductive subgroup of G. Consider the quotient space

$$\mathcal{M}_R(K) := \mathrm{Hom}(\pi_1(X, x_0), K)/K,$$

where K as before is a maximal compact subgroup of G. The inclusion of K in G produces an inclusion

$$\xi' : \mathcal{M}_R(K) \longrightarrow \mathcal{M}_R(G). \tag{3.6}$$

Corollary 3.6. *There is a natural deformation retraction of $\mathcal{M}_R(G)$ to the subset $\mathcal{M}_R(K)$ in* (3.6)*.*

Proof. The nonabelian Hodge theory gives a homeomorphism of $\mathcal{M}_R(G)$ with $\mathcal{M}_H(G)$. On the other hand, $\mathcal{M}_R(K)$ is identified with $\mathcal{M}(G)$, and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}(G) & \xrightarrow{\xi} & \mathcal{M}_H(G) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{M}_R(K) & \xrightarrow{\xi'} & \mathcal{M}_R(G) \end{array}$$

Hence Proposition 3.5 produces the deformation retraction in question. □

4 Pullback of Higgs bundles by finite morphisms

Take (X, ω) to be as before. Let M be compact connected Kähler manifold, and let

$$f : M \longrightarrow X$$

be a surjective holomorphic map such that each fiber of f is a finite subset of M. In particular, we have $\dim M = \dim X$. It is known that the form $f^*\omega$ represents a Kähler class on the Kähler manifold M [BiSu, p. 438, Lemma 2.1]. The degree of torsion-free coherent analytic sheaves on M will be defined using the Kähler class given by $f^*\omega$.

Proposition 4.1. *Let (E_G, θ) be a Higgs G–bundle on X such that the pulled back Higgs G–bundle $(f^*E_G, f^*\theta)$ on M is semistable. Then the principal G–bundle f^*E_G is semistable.*

Proof. Since the pulled back Higgs G–bundle $(f^*E_G, f^*\theta)$ is semistable, it follows that (E_G, θ) is semistable. Indeed, the pullback of any reduction of structure group of (E_G, θ) that contradicts the semistability condition also contradicts the semistability condition for $(f^*E_G, f^*\theta)$. Since the Higgs G–bundle (E_G, θ) is semistable, we conclude that the principal G–bundle E_G is semistable [Bi, p. 305, Lemma 6.2]. This, in turn, implies that f^*E_G is semistable (see [BiSu, p. 441, Theorem 2.4] and [BiSu, p. 442, Remark 2.5]). □

Proposition 4.2. *Let (E_G, θ) be a Higgs G–bundle on X such that the pulled back Higgs G–bundle $(f^*E_G, f^*\theta)$ on M is stable. Then the principal G–bundle f^*E_G is polystable.*

Proof. The principal G–Higgs bundle (E_G, θ) is stable, because any reduction of it contradicting the stability condition pulls back to a reduction that contradicts the stability condition for $(f^*E_G, f^*\theta)$. Since (E_G, θ) is stable, we know that E_G is polystable [Bi, p. 306, Lemma 6.4]. Now f^*E_G is polystable because E_G is so [BiSc, p. 439, Proposition 2.3], [BiSc, p. 442, Remark 2.6]. □

5 Co–Higgs bundles

We recall the definition of a co–Higgs vector bundle [Ra1, Ra2, Hi].

Let (X, ω) be a compact connected Kähler manifold and E a holomorphic vector bundle on X. A *co–Higgs field* on E is a holomorphic section

$$\theta \in H^0(X, \mathrm{End}(E) \otimes TX)$$

such that the section $\theta \bigwedge \theta$ of $\mathrm{End}(E) \otimes \bigwedge^2 TX$ vanishes identically. A co–Higgs bundle on X is a pair (E, θ), where E is a holomorphic vector bundle on X and θ is a co–Higgs field on E [Ra1, Ra2, Hi].

A co–Higgs bundle (E, θ) is called *semistable* if for all nonzero coherent analytic subsheaves $F \subset E$ with $\theta(F) \subset F \otimes TX$, the inequality

$$\mu(F) := \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \leq \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)} := \mu(E)$$

holds.

5.1 Co–Higgs bundles on Calabi–Yau manifolds

In this subsection we assume that $c_1(TX) \in H^2(X, \mathbb{Q})$ is zero, and the Kähler form ω on X is Ricci–flat. Take a holomorphic vector bundle E on X.

Lemma 5.1. *Let θ be a Higgs field or a co–Higgs field on E such that (E, θ) is semistable. Then the vector bundle E is semistable.*

Proof. Let θ be a co–Higgs field on E such that the co–Higgs bundle (E, θ) is semistable. Assume that E is not semistable. Let F be the maximal semistable subsheaf of E, in other words, F is the first term in the Harder–Narasimhan filtration of E. The maximal semistable subsheaf of E/F will be denoted by F_1, so $\mu_{\max}(E/F) = \mu(F_1)$. Note that we have

$$\mu(F) > \mu(F_1) = \mu_{\max}(E/F)\,. \tag{5.1}$$

Since ω is Ricci–flat we know that TX is polystable. The tensor product of a semistable sheaf and a semistable vector bundle is semistable [AB, p. 212, Lemma 2.7]. Therefore, the maximal semistable subsheaf of $(E/F) \otimes TX$ is

$$F_1 \otimes TX \subset (E/F) \otimes TX\,.$$

Now,

$$\mu(F_1 \otimes TX) = \mu(F_1)$$

because $c_1(TX) = 0$. Hence from (5.1) it follows that

$$\mu(F) > \mu(F_1 \otimes TX) = \mu_{\max}((E/F) \otimes TX)\,. \tag{5.2}$$

Let

$$q : E \longrightarrow E/F$$

be the quotient homomorphism. From (5.2) it follows that there is no nonzero homomorphism from E to $(E/F) \otimes TX$. In particular, the composition

$$F \hookrightarrow E \xrightarrow{\theta} E \otimes TX \xrightarrow{q \otimes \mathrm{Id}} (E/F) \otimes TX$$

vanishes identically. This immediately implies that $\theta(F) \subset F \otimes TX$. Therefore, the co–Higgs subsheaf $(F, \theta|_F)$ of (E, θ) violates the inequality in the definition of semistability. But this contradicts the given condition that (E, θ) is semistable. Hence we conclude that E is semistable.

Note that Ω_X is polystable because TX is polystable. Hence the above proof also works when the co-Higgs field θ is replaced by a Higgs field. □

A particular case of this result was shown in [Ra2] for X a K3 surface. Moreover, a result implying this Lemma was proved in [BH].

5.2 A characterization of Calabi–Yau manifolds

Theorem 5.2. *Let X be a compact connected Kähler manifold such that for every Kähler class $[\omega] \in H^2(X, \mathbb{R})$ on it the following two hold:*

1. *the tangent bundle TX is semistable, and*
2. *for every semistable Higgs or co–Higgs bundle (E, θ) on X, the underlying holomorphic vector bundle E is semistable.*

Then $c_1(TX) = 0$.

Proof. We will show that degree$(TX) = 0$ for every Kähler class on X. For this, take any Kähler class $[\omega]$.

First assume that degree$(TX) > 0$ with respect to $[\omega]$. We will construct a co–Higgs field on the holomorphic vector bundle

$$E := \mathcal{O}_X \oplus TX\,. \tag{5.3}$$

Since the vector bundle $\mathrm{Hom}(TX, \mathcal{O}_X) = \Omega_X$ is a direct summand End(E), we have

$$\mathrm{End}(TX) = \Omega_X \otimes TX = \mathrm{Hom}(TX, \mathcal{O}_X) \otimes TX \subset \mathrm{End}(E) \otimes TX\,.$$

Hence $\mathrm{Id}_{TX} \in H^0(X, \mathrm{End}(TX))$ is a co–Higgs field on E; this co–Higgs field on E will be denoted by θ.

We will show that the co–Higgs bundle (E, θ) is semistable.

For show that, take any coherent analytic subsheaf $F \subset E$ such that $\theta(F) \subset F \otimes TX$. First consider the case where

$$F \bigcap (0, TX) = 0\,.$$

Then the composition

$$F \hookrightarrow E = \mathcal{O}_X \oplus TX \longrightarrow \mathcal{O}_X$$

is injective. Hence

$$\mu(F) \leq \mu(\mathcal{O}_X) = 0 < \mu(E)\,.$$

Hence the co–Higgs subsheaf $(F, \theta|_F)$ of (E, θ) does not violate the inequality condition for semistability.

Next assume that

$$F \bigcap (0, TX) \neq 0\,.$$

Now in view of the given condition that $\theta(F) \subset F \otimes TX$, from the construction of the co–Higgs field θ is follows immediately that

$$F \bigcap (\mathcal{O}_X,\, 0) \neq 0\,.$$

Hence we have

$$F = (F \bigcap (0,\, TX)) \oplus (F \bigcap (\mathcal{O}_X,\, 0))\,. \tag{5.4}$$

Note that

$$\mu(F \bigcap (0,\, TX)) \leq \mu(TX)$$

because TX is semistable, and also we have $\mu(F \bigcap (\mathcal{O}_X,\, 0)) \leq \mu(\mathcal{O}_X)$. Therefore, from (5.4) it follows that

$$\mu(F) \leq \mu(E)\,.$$

Hence again the co–Higgs subsheaf $(F,\, \theta|_F)$ of $(E,\, \theta)$ does not violate the inequality condition for semistability. So $(E,\, \theta)$ is semistable.

Hence by the given condition, the holomorphic vector bundle E is semistable. But this implies that degree$(TX) = 0$. This contradicts the assumption that degree$(TX) > 0$.

Now assume that degree$(TX) < 0$. We will construct a Higgs field on the vector bundle E in (5.3).

The vector bundle $\mathrm{Hom}(\mathcal{O}_X,\, TX) = TX$ is a direct summand $\mathrm{End}(E)$. Hence we have

$$\mathrm{End}(TX) = TX \otimes \Omega_X = \mathrm{Hom}(\mathcal{O}_X,\, TX) \otimes \Omega_X \subset \mathrm{End}(E) \otimes \Omega_X\,.$$

Consequently, $\mathrm{Id}_{TX} \in H^0(X,\, \mathrm{End}(TX))$ is a Higgs field on E; this Higgs field on E will be denoted by θ'.

We will show that the above Higgs vector bundle $(E,\, \theta)$ is semistable.

Take any coherent analytic subsheaf

$$F \subset E$$

such that $\theta(F) \subset F \otimes \Omega_X$ and $\mathrm{rank}(F) < \mathrm{rank}(E)$. First consider the case where

$$F \bigcap (\mathcal{O}_X,\, 0) = 0\,.$$

Then we have $F \subset (0,\, TX) \subset E$. Since TX is semistable, we have

$$\mu(F) \leq \mu(TX)\,.$$

On the other hand, $\mu(TX) < \mu(E)$, because degree$(TX) < 0 = \mu(\mathcal{O}_X)$. Combining these we get

$$\mu(F) < \mu(E)\,,$$

and consequently, the Higgs subsheaf $(F, \theta|_F)$ of (E, θ) does not violate the inequality condition for semistability.

Now assume that

$$F \bigcap (\mathcal{O}_X, 0) \neq 0 .$$

Hence

$$\text{rank}(F \bigcap (\mathcal{O}_X, 0)) = 1 , \tag{5.5}$$

because $F \bigcap (\mathcal{O}_X, 0)$ is a nonzero subsheaf of $\mathcal{O}_X$. Now from the construction of the Higgs field θ it follows that

$$\text{rank}(F \bigcap (0, TX)) = \text{rank}(TX) .$$

Combining this with (5.5) we conclude that $\text{rank}(F) = \text{rank}(E)$. This contradicts the assumption that $\text{rank}(F) < \text{rank}(E)$. Hence we conclude that the Higgs vector bundle (E, θ) is semistable.

Now the given condition says that E is semistable, which in turn implies that

$$\text{degree}(TX) = 0 .$$

This contradicts the assumption that $\text{degree}(TX) < 0$.

Therefore, we conclude that $\text{degree}(TX) = 0$ for all Kähler classes $[\omega]$ on X. In other words,

$$c_1(TX) \cup ([\omega])^{d-1} = 0 \tag{5.6}$$

for every Kähler class $[\omega]$ on X, where d as before is the complex dimension of d. But the $\mathbb{R}$–linear span of

$$\{[\omega]^{d-1} \in H^{2d-2}(X, \mathbb{R}) \mid [\omega] \quad \text{Kähler class}\}$$

is the full $H^{2d-2}(X, \mathbb{R})$. Therefore, from (5.6) it follows that

$$c_1(TX) \cup \delta = 0$$

for all $\delta \in H^{2d-2}(X, \mathbb{R})$. Now from the Poincaré duality it follows that $c_1(TX) \in H^2(X, \mathbb{R})$ vanishes. □

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Indranil Biswas
School of Mathematics, Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai 400005, India
indranil@math.tifr.res.in

Ugo Bruzzo
Scuola Internazionale Superiore di Studi Avanzati (SISSA)
Via Bonomea 265, 34136 Trieste, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste
bruzzo@sissa.it

Beatriz Graña Otero
Departamento de Matemáticas, Pontificia Universidad Javeriana,
Cra. 7$^{\text{ma}}$ N$^{\text{o}}$ 40-62, Bogotá, Colombia
bgrana@javeriana.edu.co

Alessio Lo Giudice
Via Antonio Locatelli 20, 37122 Verona
alessiologiudic@gmail.com